Terms and concepts covered: Deck transformations, regular covers.

Corresponding reading: Hatcher Ch 1.3

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Prove that a covering map $p : \tilde{X} \to X$ is an open map, that is, the image of an open subset is open.

2. **Definition (local homeomorphism).** A continuous map $f : X \to Y$ is a *local homeomorphism* if every point $x \in X$ has a neighbourhood U such that $f(U) \subseteq Y$ is open, and the restriction $f|_U : U \to f(U)$ is a homeomorphism.

   Definition (locally homeomorphic). A space X is *locally homeomorphic* to a space Y if every point in X has an open neighbourhood homeomorphic to an open subset of Y.

   Note that this definition is not symmetric in X and Y.

   (a) Show that if there exists a local homeomorphism $X \to Y$, then X is locally homeomorphic to Y. The converse is not true, for example, S^2 and $\mathbb{R}^2$ are locally homeomorphic to each other, but no local homeomorphism exists in either direction.

   (b) Verify that a covering map $p : \tilde{X} \to X$ is a local homeomorphism.

   (c) Verify that a local homeomorphism $f : X \to Y$ preserves local properties. For example, X will satisfy each of the following properties if and only if $f(X)$ does.

      (i) local connectedness and local path-connectedness
      (ii) local compactness
      (iii) first countability (every point has a countable neighbourhood basis)
      (iv) being locally Euclidean

3. Let $p : E \to B$ be a covering map. We say that a open subset U in B is *evenly covered* by p if $p^{-1}(U)$ is the disjoint union of open subsets $\{U_\alpha\}$ such that for each α the covering map $p|_{U_\alpha}$ restricts to a homeomorhpism from U_α to U.

   (a) Verify that the definition of a covering map p is the statement that every point $b \in B$ has a neighbourhood evenly covered by p.

   (b) Show that, if an open subset $U \subseteq B$ is evenly covered by p, then any open subset $V \subseteq U$ is evenly covered by p.

   (c) Deduce that, if p is a covering map, every point has a neighbourhood basis of open sets that are evenly covered by p.

4. Suppose that X is a connected space. Show that the only subsets of X that are both open and closed are X and $\varnothing$.

5. Prove that a 1-sheeted cover is a homeomorphism.

6. Let X be path-connected, locally path-connected semi-locally simply-connected space. Explain why, if X is simply connected, the only covers of X are homeomorphisms $X \to X$.

7. Let X, Y be path-connected, locally path-connected spaces. Assume Y is semi-locally simply connected. Given a map $f : (X, x_0) \to (Y, y_0)$, which path-connected covers $\tilde{Y}$ of Y will f lift to?

8. Let H be a subgroup of a group G.

   (a) Define the *normalizer* $N_G(H)$ of H in G.

(b) Show that $N_G(H)$ is a subgroup of G.

(c) Show that H is contained in $N_G(H)$, and is normal in $N_G(H)$.

(d) Show that if H is a normal subgroup of G, then $N_G(H) = G$.

(e) Show that $N_G(H)$ is maximal in the following sense: if J is a subgroup $H \subseteq J \subseteq G$ and H is normal in J, then $J \subseteq N_G(H)$.

9. Let $p : \tilde{X} \to X$ be the covering map of a connected cover, and let $\tau : \tilde{X} \to \tilde{X}$ be a deck transformation. Prove that if τ fixes a point, then τ is the identity map.

10. We have illustrated a cover $\tilde{X}$ of $S^1 \vee S^1$ as a graph where every edge is directed and labelled by either a or b. Explain how this convention encodes the data of the covering map, and explain why a deck transformation of $\tilde{X}$ is precisely a graph automorphism that preserves the direction and label of every edge.

11. (a) Suppose that X has an abelian fundamental group. Explain why every cover of X is regular.

    (b) Explain why every 2-sheeted cover is regular.

12. Let G be a group acting on a set X.

    (a) Define *orbit*. Prove that the condition "x_1 and x_2 are in the same orbit" defines an equivalence relation on X.

    (b) State and prove the orbit-stabilizer theorem.

13. **Definition (Group acting on a space, I).** Let G be a group. A group action of G on a space Y is a group homomorphism $\rho : G \to \mathrm{Homeo}(Y)$, where $\mathrm{Homeo}(Y)$ is the group of homeomorhpisms $Y \to Y$.

    Note: If G were a topological group, we would want to impose extra conditions on our group action to be compatible with the topology. Here we assume that G has no topology, or, equivalently, we may assume that G has the discrete topology.

    Verify that the definition of a group action on a space is equivalent to the following.

    Definition (Group acting on a space, II). Let G be a group. A group action of G on a space Y is a map

$$\begin{aligned} \alpha : G \times Y &\longrightarrow Y \\ (g, y) &\longmapsto g \cdot y \end{aligned}$$

    satisfying three conditions.

    (i) For each fixed $g \in G$, the coresponding map is continuous:

$$\begin{aligned} g : Y &\longrightarrow Y \\ y &\longmapsto g \cdot y \end{aligned}$$

    (ii) For each $g, h \in G$ and $y \in Y$, $(gh) \cdot y = g \cdot (h \cdot y)$.

    (iii) For $e \in G$ the identity, $e \cdot y = y$ for all $y \in Y$.

14. (a) Let G be a group with a covering space action on a space Y (Assignment Problem 5). Prove that the action is free.

    (b) Show by example that a free action of a group G on a space Y need not be a covering action. *Hint:* $\mathbb{R}$ acts on $\mathbb{R}$.

Assignment questions

(Hand these questions in!)

1. (a) Let F_n be the free group on n generators, and F_m the free group on m generators. Show that, if $F_m \cong F_n$, the $m = n$. Conclude that the number n (called the *rank* of F_n) is an isomorphism invariant. *Hint:* abelianization.

   (b) Let X be a connected, finite graph with n vertices and e edges. Show that $\pi_1(X)$ is the free group of rank $(e - n + 1)$. You may assume the following result from graph theory.

   Proposition (Combinatorics of trees). Let T be a finite tree (which is by definition connected). If T has n vertices, then it has $(n - 1)$ edges.

   (c) Let X be a connected m-sheeted cover of the wedge $\bigvee_n S^1$ of n circles. We proved on Homework 6 that $\pi_1(X)$ is a free group. What is the rank of $\pi_1(X)$?

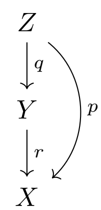
   (d) Prove the following theorem.

   Theorem (Schreier index formula). Let F_n be the free group of rank n. A subgroup of index $m \in \mathbb{N}$ in F_n has rank $1 + m(n - 1)$. An infinite-rank subgroup has infinite index.

   This theorem shows that the rank of a finite-index subgroup is a function of its index. Moreover, the larger the index (so "smaller" the subgroup), the larger its rank!

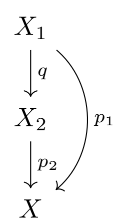
2. **(The Galois correspondence for covering spaces).** In this question, we assume all spaces are path-connected, locally path-connected, and semi-locally simply-connected.

   (a) Let p, q, r be continuous maps with $p = r \circ q$. Show that, if p and r are covering maps, then so is q.



   (b) Suppose that H_1, H_2 are subgroups of the fundamental group $\pi_1(X, x_0)$ of a space X, and let $p_1 : (X_1, x_1) \to (X, x_0)$ and $p_2 : (X_2, x_2) \to (X, x_0)$ be the covering spaces such that $(p_1)_*$ and $(p_2)_*$ induce the inclusions of H_1 and H_2, respectively, into $\pi_1(X, x_0)$. (Recall from our classification of covering spaces in Homework 6 Problem #4 that these covering spaces exist and are unique).

   Explain why p_1 factors through p_2 (as in the diagram below) if and only if $H_1 \subseteq H_2$.



   Conclude from part (a) that, if it factors, the map q is a covering map.

   (c) Give a precise statement of the resulting strengthening of our classification theorem for based covering spaces of X: for every subgroup of $\pi_1(X, x_0)$ there is a unique covering space, and for every inclusion of subgroups $H_1 \to H_2$ there is an intermediate covering map.

   Remark: There is, in fact, an isomorphism of posets between the subgroups of $\pi_1(X)$ (ordered by inclusion) and the covers of X (ordered by existence of intermediate covers). For a more detailed statement, see Hatcher Chapter 1.3 Problem 24. This result is sometimes called the *Galois correspondence* for covering spaces, in analogy to the Galois correspondence for field extensions.

   (d) Let $X = \mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$. Draw the system of based covering maps of X and intermediate covers, and label the fundamental group of each space.

3. **(Topology QR Exam, September 2016).** Give an example of a (connected) 3-fold covering which is not regular.

4. Let Σ_g be the genus-g surface for $g \geq 1$. Prove that Σ_g has a regular n-sheeted connected cover for every $n \geq 1$.

5. **(Covering spaces as quotients by covering actions).** You may refer to Hatcher p72-73 while you write your solution to this problem.

   (a) **Definition (Orbit space).** Let G be a group acting on a space Y. Recall that the *orbit* of a point $y \in Y$ is the subset
   $$G \cdot y = \{g \cdot y \mid g \in G\} \subseteq Y.$$
   The *orbit space* of this action, denoted Y/G, is the quotient space of Y in which every orbit is identified to a point.

   Let $p : \tilde{X} \to X$ be a (surjective) normal covering space with deck group $G(\tilde{X})$. Verify that we can identify X with the orbit space $\tilde{X}/G(\tilde{X})$, and $p : \tilde{X} \to X$ with the quotient map.

   (b) **Definition (Covering space action).** Let G be a group acting on a space Y. Then this action is *covering space action* if it satisfies the following condition. Each $y \in Y$ has a neighbourhood U such that all images $g(U)$ for distinct $g \in G$ are disjoint. In other words, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$.

   Let $p : \tilde{X} \to X$ be a connected covering space. Verify that the action of the deck group $G(\tilde{X})$ is a covering space action.

   (c) Now suppose that a group G is acting on a space Y by a covering space action. Prove that the quotient map $p : Y \to Y/G$ is a normal covering space.

   (d) Let G is acting on a space Y by a covering space action, and suppose Y is path-connected. Prove that the Deck group of the cover $p : Y \to Y/G$ is isomorphic to G.

   (e) Let G is acting on a space Y by a covering space action, and suppose Y is path-connected and locally path-connected. Let $p : Y \to Y/G$ be the quotient. Prove that
   $$\frac{\pi_1(Y/G, G \cdot y_0)}{p_*(\pi_1(Y, y_0))} \cong G.$$
   In particular, if Y is simply-connected, then $\pi_1(Y/G) \cong G$.

   (f) On Homework 5 Problem 2(f), you constructed the covers of the torus associated to the subgroups $4\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 4\mathbb{Z}$, and $2\mathbb{Z} \times 2\mathbb{Z}$ of its fundamental group $\mathbb{Z}^2$. Explain how you could construct these covering spaces using a suitable action of the groups $4\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 4\mathbb{Z}$, and $2\mathbb{Z} \times 2\mathbb{Z}$, respectively, on the universal cover $\mathbb{R}^2$ of the torus.

   (g) Suppose we have a covering space action of a group G on a simply connected space Y. Let $H_1 \subseteq H_2 \subseteq G$ be subgroups. Explain how to use the action of H_1, H_2 on Y to construct the intermediate cover $q : X_1 \to X_2$ defined in Assignment Problem 2 (b). What happens in the special cases $H_1 = 0$ or $H_2 = G$? You do not need to check details.

6. **(Lens space).**

   (a) Prove that any free action of a finite group on a Hausdorff space Y is a covering space action.

   (b) Let $S^3 \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$ be the unit sphere. For coprime integers p, q, define an action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 by
   $$(z_1, z_2) \longmapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2).$$
   Verify that this action is free.

   (c) The orbit space $S^3 \,/\, \mathbb{Z}/p\mathbb{Z}$ is a 3-manifold called a *lens space*. What is its fundamental group? *Hint:* Assignment Problem 5.

7. **(Bonus).** Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering map for path-connected, locally path-connected spaces $\tilde{X}, X$. Let $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, and let $N(H)$ be its normalizer in $\pi_1(X, x_0)$. We proved that $N(H)$ acts on $\tilde{X}$ by Deck transformations, and it acts on the fibre $p^{-1}(x_0)$ by permutations. What is the relationship between these actions?

Wellbeing

(This section is completely optional. This is a nudge to prioritize your wellbeing during the pandemic.)

1. **(Health comes first).** Make your health and happiness goals a priority this week.
2. **(Savouring).** Every day this week, choose an experience to savour. This might be a tasty meal, a walk outside, a warm bath or shower, a self-massage, a favourite song, a scented lotion, or a hot cup of tea or coffee. Be present in the moment. Focus on the sensory experiences, and what you enjoy about them. Think about what elements of the experience you will want to remember.