Final Exam

Math 592
28 April 2021
Jenny Wilson

Name: ______________________________

Instructions: This exam has 4 questions for a total of 35 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 60 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	4	
2	4	
3	15	
4	12	
Total:	35	

Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (unit n-sphere) (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

1. (4 points) A CW complex X consists of

   - two vertices x and y
   - three edges a, b, c, where a is a directed edge from x to y, b is a directed edge from y to x, and c is a directed edge from y to y,
   - two 2-cells A and B, where A is glued along the word ac^2b and B is glued along the word $bac^{-1}ba$.

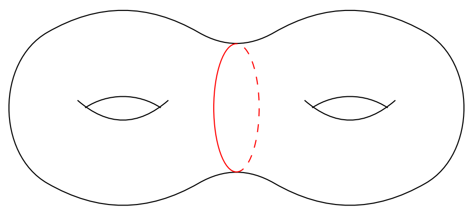
   Compute the homology groups of X.

2. (4 points) Let F_3 be the free group on 3 letters. Prove that every finite-index subgroup of F_3 is a free group of odd rank, and that every free group of odd rank at least 3 occurs as a finite-index subgroup of F_3.

3. Consider the following descriptions of hypothetical maps f. In each part, prove that no such continuous map f exists.

   (a) (2 points) The map $f : S^4 \to \mathbb{C}\mathrm{P}^2$ is a homotopy equivalence.

   (b) (3 points) Let Σ_2 be the closed genus-2 surface. The map $f : \Sigma_2 \to A$ is a retraction onto the circle $A \subseteq \Sigma_2$ shown below. (Graphics credit: Salman Siddiqi)



(c) (2 points) Fix $n \geq 1$. The map $f : S^n \to S^n$ is a map of degree 2 with no fixed point.

(d) (3 points) Let T be the torus. The map $f : S^2 \to T$ is an isomorphism on degree-2 homology.

(e) (2 points) Let X be a finite CW complex, and $n \geq 1$. The map $f : \mathbb{R}\mathrm{P}^{2n} \to X$ is a d-sheeted covering map for some $d > 1$.

(f) (3 points) Let X be a space, and $n \geq 1$. The map $f : \mathbb{C}\mathrm{P}^{n} \to X$ is a covering map which is nullhomotopic.

4. (12 points) For each of the following statements: if the statement is true, write "`True`". If not, state a counterexample. **No justification necessary.**
Note: If the statement is false, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let X, Y be spaces, and $A \subseteq X$ a subspace. Suppose $f : X \to Y$ is a homotopy equivalence. Then $f|_A : A \to f(A)$ is a homotopy equivalence.

   (b) If A is a retract of X (not necessarily a deformation retract), then A and X are homotopy equivalent.

   (c) Let $A \subseteq X$. If A is a deformation retract of X, then $H_n(X, A) = 0$ for all n.

   (d) Let F be a covariant functor from the category of topological spaces and continuous maps, to the category of abelian groups and group homomorphisms. If f is a homeomorphism of topological spaces, then $F(f)$ is an isomorphism of abelian groups.

(e) If $f : S^n \to S^n$ is not surjective, then it is nullhomotopic.

(f) If $f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ is not surjective, then it is nullhomotopic.

(g) Suppose a space X is a union $X = U \cup V$ of contractible open subsets U and V. Then $\widetilde{H}_n(X) \cong \widetilde{H}_{n-1}(U \cap V)$ for all n.

(h) Let d be the local degree of a map $f : S^n \to S^n$ at a point $x \in S^n$. Then d must be ± 1.

(i) Let M be a closed, smoothly embedded submanifold of $\mathbb{R}^n$, and let $\mathbb{R}^n/M$ be the quotient collapsing M to a point. Then $\widetilde{H}_k(\mathbb{R}^n/M) \cong \widetilde{H}_{k-1}(M)$ in each degree k.

(j) There is no path-connected space X with universal cover $\tilde{X}$ satisfying $H_1(\tilde{X}) = \mathbb{Z}^2$.

(k) There is no path-connected space X with universal cover $\tilde{X}$ satisfying $H_2(\tilde{X}) = \mathbb{Z}^2$.

(l) Let X be a simply connected, finite CW complex. If $f : X \to X$ is homotopic to the identity, it must have a fixed point.