Final Exam

Math 592
28 April 2021
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Name: ______________________________

Instructions: This exam has 4 questions for a total of 35 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 60 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	4	
2	4	
3	15	
4	12	
Total:	35	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
  (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

1. (4 points) A CW complex X consists of

- two vertices x and y
- three edges a, b, c, where a is a directed edge from x to y, b is a directed edge from y to x, and c is a directed edge from y to y,
- two 2-cells A and B, where A is glued along the word ac^2b and B is glued along the word $bac^{-1}ba$.

Compute the homology groups of X.

Solution. The cellular chain groups of X are

$$C_n(X) = 0 \text{ for } n \geq 3, \quad C_2(X) = \mathbb{Z}\{A, B\}, \quad C_1(X) = \mathbb{Z}\{a, b, c\}, \quad C_0(X) = \mathbb{Z}\{x, y\}.$$

Thus our boundary maps are

$$\begin{aligned} \partial_2 : C_2(X) &\longrightarrow C_1(X) \\ \mathbb{Z}\{A, B\} &\longrightarrow \mathbb{Z}\{a, b, c\} \\ A &\longmapsto a + b + 2c \\ B &\longmapsto 2a + 2b - c \end{aligned} \qquad \begin{aligned} \partial_1 : C_1(X) &\longrightarrow C_2(X) \\ \mathbb{Z}\{a, b, c\} &\longrightarrow \mathbb{Z}\{x, y\} \\ a &\longmapsto y - x \\ b &\longmapsto x - y \\ c &\longmapsto y - y = 0 \end{aligned}$$

Then our chain complex is

$$\longrightarrow 0 \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

$$\ker(\partial_2) = 0 \qquad \ker(\partial_1) = \mathbb{Z}\{a + b, c\} \qquad \ker(\partial_0) = \mathbb{Z}\{x, y\}$$

$$\operatorname{im}(\partial_3) = 0 \qquad \operatorname{im}(\partial_2) = \mathbb{Z}\left\{ \begin{array}{c} a + b + 2c, \\ 2a + 2b - c \end{array} \right\} \qquad \operatorname{im}(\partial_1) = \mathbb{Z}(y - x)$$

We conclude (by performing changes-of-bases) that

$$H_2(X) = \frac{0}{0} = 0 \qquad H_1(X) = \frac{\mathbb{Z}\{a + b + 2c, c\}}{\mathbb{Z}\{a + b + 2c, 5c\}} \cong \mathbb{Z}/5\mathbb{Z} \qquad H_0(X) = \frac{\mathbb{Z}\{x, y - x\}}{\mathbb{Z}\{y - x\}} \cong \mathbb{Z}.$$

Alternate approach: compute the Smith normal form of the differentials and use our formula for homology

$$\partial_2 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & -1 \end{bmatrix}, \ \mathrm{SNF}(\partial_2) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \partial_1 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \ \mathrm{SNF}(\partial_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. (4 points) Let F_3 be the free group on 3 letters. Prove that every finite-index subgroup of F_3 is a free group of odd rank, and that every free group of odd rank at least 3 occurs as a finite-index subgroup of F_3.

Solution. We can identify F_3 with the fundamental group of the wedge $X = S^1 \vee S^1 \vee S^1$. The space X is path-connected, and since it is a graph it is locally path-connected and semi-locally simply-connected. Hence, by our classification of covering spaces, each subgroup G of F_3 is isomorphic to the fundamental group of a path-connected cover of X. We proved that the index of the subgroup equals the number of sheets of the cover, so in particular the finite-index subgroups G of F_3 correspond to finite-sheeted covers of X.

Let $\tilde{X} \to X$ be a d-sheeted cover of X. We proved that a cover $\tilde{X}$ of a graph X is itself a graph, and that if X has 1 vertex and 3 edges, then $\tilde{X}$ has d vertices and $3d$ edges. A spanning tree in $\tilde{X}$ will contain all d vertices, and $(d-1)$ edges. We proved that the fundamental group of a graph is free, with a free generator for each edge not contained in our chosen spanning tree. Thus $\pi_1(\tilde{X})$ is the free group of rank

$$3d - (d-1) = 2d + 1.$$

This rank is always odd, and as d ranges through the positive integers, $2d+1$ ranges through all odd integers 3 and above.

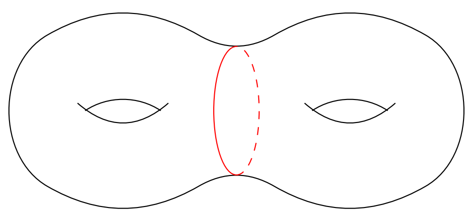
To complete the problem, we must show that X has a d–sheeted cover for every $d \geq 1$. The identity map $X \to X$ is a one-sheeted cover. For $d > 1$, consider a surjective homomorphism ϕ_d from F_3 to the cyclic group C_d of order d (say, a map sending all three free generators of F_3 to any generator of C_d.) Then the kernel G of ϕ_d is an index-d subgroup, and so corresponds to a d-sheeted cover, and we conclude that $G \cong F_{2d+1}$.

3. Consider the following descriptions of hypothetical maps f. In each part, prove that no such continuous map f exists.

   (a) (2 points) The map $f : S^4 \to \mathbb{C}\mathrm{P}^2$ is a homotopy equivalence.

   Solution. If f were a homotopy equivalence, then for each n the induced map $f_* : H_n(S^4) \to H_n(\mathbb{C}\mathrm{P}^2)$ would be an isomorphism. However, we calculated that $H_2(S^4) = 0$ and $H_2(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}$.

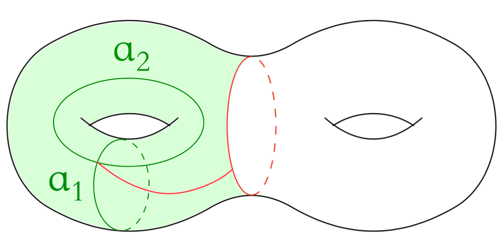
   (b) (3 points) Let Σ_2 be the closed genus-2 surface. The map $f : \Sigma_2 \to A$ is a retraction onto the circle $A \subseteq \Sigma_2$ shown below. (Graphics credit: Salman Siddiqi)

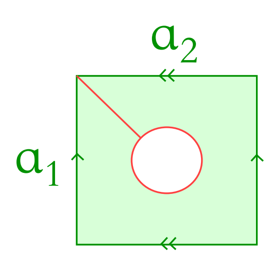


   Solution. Let $\iota : A \to \Sigma_2$ be the inclusion of A. If $f : \Sigma_2 \to A$ were a retraction, then by definition $f \circ \iota = id_A$, and by functoriality of homology the composite

$$H_1(A) \xrightarrow{\iota_*} H_1(\Sigma_2) \xrightarrow{f_*} H_1(A)$$

   would be the identity map on the group $H_1(A) = H_1(S^1) \cong \mathbb{Z}$. This is impossible, however, because $\iota_* : H_1(A) \to H_1(\Sigma_2)$ is the zero map. We can see this, for example, since we see in the diagrams below that A (identified with an element of $\pi_1(\Sigma_2)$) is the commutator $[a_1, a_2]$ of the loops a_1, a_2, hence vanishes in the abelianization $H_1(\Sigma_2)$ of $\pi_1(\Sigma_2)$.





(c) (2 points) Fix $n \geq 1$. The map $f : S^n \to S^n$ is a map of degree 2 with no fixed point.

Solution. The Lefschetz fixed point theorem states that a self-map of a finite CW complex can have no fixed points only if its Lefschetz number is zero. But f has Lefschetz number

$$\begin{aligned}\tau(f) &= \sum_{k=0}^{n} (-1)^k \text{ Trace}\{f_* : H_k(S^n) \to H_k(S^n)\} \\ &= (-1)^0(1) + (-1)^n(2) \\ &= 1 \pm 2 \neq 0\end{aligned}$$

and therefore must have a fixed point.

Alternative Solution. We showed that if f has no fixed points, then

$$f_t(x) = \frac{(1-t)f(x) - tx}{||(1-t)f(x) - tx||}$$

is a homotopy from f to the antipodal map. Thus f has degree $(-1)^{n+1} \neq 2$.

(d) (3 points) Let T be the torus. The map $f : S^2 \to T$ is an isomorphism on degree-2 homology.

Solution. In brief: the map f lifts to the (contractible) universal cover $\mathbb{R}^2$ of T, so f induces the zero map on degree-2 homology (in fact, f is nullhomotopic).

Let us verify the details. We showed that the universal cover of the torus is the plane $\mathbb{R}^2$, so let $p : \mathbb{R}^2 \to T$ denote this covering map. The sphere S^2 is path-connected and locally path-connected. Since S^2 has trivial fundamental group, for $x \in S^2$ and $\tilde{y} \in p^{-1}(f(x))$ we must have containment

$$f_*(\pi_1(S^2, x)) = 0 \subseteq p_*(\pi_1(\mathbb{R}^2, \tilde{y})).$$

Thus our lifting criterion for covers implies that a lift $\tilde{f} : S^2 \to T$ exists, that is, the map f factors $f = p \circ \tilde{f}$.

By functoriality of homology, then, the induced map f_* on degree-2 homology factors as a composite of maps

$$H_2(S^2) \xrightarrow{\tilde{f}_*} H_2(\mathbb{R}^2) \xrightarrow{p_*} H_2(T)$$

But $H_2(\mathbb{R}^2) = 0$ and $H_2(S^2) \cong H_2(T) \cong \mathbb{Z}$, so f_* cannot be an isomorphism.

(e) (2 points) Let X be a finite CW complex, and $n \geq 1$. The map $f : \mathbb{RP}^{2n} \to X$ is a d-sheeted covering map for some $d > 1$.

Solution. Since f is a covering map, we proved that the Euler characteristics of $\mathbb{RP}^{2n}$ and X must satisfy

$$\chi(\mathbb{RP}^{2n}) = d\,\chi(X)$$

But we computed $\chi(\mathbb{RP}^{2n}) = 1$, so d must be 1.

(f) (3 points) Let X be a space, and $n \geq 1$. The map $f : \mathbb{CP}^n \to X$ is a covering map which is nullhomotopic.

Solution. We note that the space $\mathbb{CP}^n$ is connected but not contractible; non-contractibility follows (for example) since it has nonzero homology in degree $2n$. Thus the result follows from the following lemma.

> **Lemma.** Let $p : \tilde{X} \to X$ be a covering map with $\tilde{X}$ connected. Then p is not nullhomotopic unless $\tilde{X}$ is contractible.

Proof: Suppose $p : \tilde{X} \to X$ is a covering map. We have a commutative diagram

$$\begin{array}{ccc} & & \tilde{X} \\ & \overset{id_{\tilde{X}}}{\nearrow} & \downarrow p \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

Let $F_t : \tilde{X} \to X$ be a homotopy from p to a constant map at some point $x_0 \in X$. By the homotopy lifting property of the covering map p, there is a lift $\tilde{F}_t$.

$$\begin{array}{ccc} & & \tilde{X} \\ & \overset{\tilde{F}_t}{\nearrow} & \downarrow p \\ \tilde{X} \times I & \xrightarrow{F_t} & X \end{array}$$

Since $F_1(\tilde{X}) = \{x_0\}$, $\tilde{F}_1(\tilde{X})$ must be contained in the fibre $p^{-1}(x_0)$. But $\tilde{X}$ is connected and (by definition of a covering map) the fibre $p^{-1}(x_0)$ is discrete, so $\tilde{F}_1$ must be a constant map. It follows that $\tilde{F}_t : \tilde{X} \to \tilde{X}$ is a homotopy from the identity to a constant map. Thus $\tilde{X}$ is contractible.

4. (12 points) For each of the following statements: if the statement is true, write "`True`". If not, state a counterexample. **No justification necessary.**
Note: If the statement is false, you can receive partial credit for writing "`False`" without a counterexample.

(a) Let X, Y be spaces, and $A \subseteq X$ a subspace. Suppose $f : X \to Y$ is a homotopy equivalence. Then $f|_A : A \to f(A)$ is a homotopy equivalence.

False. The analogous statement is true for homeomorphisms, but not homotopy equivalences. Consider, for example, the constant map $f : \mathbb{R}^2 \to \{*\}$. Then f is a homotopy equivalence, but if we restrict it to some non-contractible subspace of $\mathbb{R}^2$ such as the unit circle $A = S^1$, the restriction $f|_A$ is not a homotopy equivalence.

(b) If A is a retract of X (not necessarily a deformation retract), then A and X are homotopy equivalent.

False. For example, any point $x \in X$ is a retract of X, but not every space X is contractible. For example, take X to be the unit circle S^1 in $\mathbb{C}$, so the constant map $f : S^1 \to \{1\}$ is a retract, but $X = S^1$ and $A = \{1\}$ are not homotopy equivalent.

(c) Let $A \subseteq X$. If A is a deformation retract of X, then $H_n(X, A) = 0$ for all n.

True. *Hint:* If A is a deformation retract of X, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence and so induces isomorphisms on reduced homology groups. The result then follows from the long exact sequence of a pair.

(d) Let F be a covariant functor from the category of topological spaces and continuous maps, to the category of abelian groups and group homomorphisms. If f is a homeomorphism of topological spaces, then $F(f)$ is an isomorphism of abelian groups.

True. *Hint:* Define an isomorphism $f : X \to Y$ in a category $\mathscr{C}$ by the existence of an inverse morphism $f^{-1} : Y \to X$ in $\mathscr{C}$ satisfying $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$. Then by definition of functoriality, all functors must map isomorphisms to isomorphisms.

(e) If $f : S^n \to S^n$ is not surjective, then it is nullhomotopic.

True. *Hint:* If there is a point $x \in S^n$ not in the image of f, then f factors through a map

$$S^n \to S^n \setminus \{x\} \to S^n.$$

Since $S^n \setminus \{x\} \cong D^n$ is contractible, f is nullhomotopic.

(f) If $f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ is not surjective, then it is nullhomotopic.

False. For example, consider the map

$$\begin{aligned} f : \mathbb{R}^{n+1} \setminus \{0\} &\longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \\ x &\longmapsto \frac{x}{||x||} \end{aligned}$$

with image the unit sphere $S^n \subsetneq \mathbb{R}^{n+1} \setminus \{0\}$. The map is f is homotopic to the identity map via the straight-line homotopy. Since S^n is not contractible, the identity map is not nullhomotopic.

(g) Suppose a space X is a union $X = U \cup V$ of contractible open subsets U and V. Then $\widetilde{H}_n(X) \cong \widetilde{H}_{n-1}(U \cap V)$ for all n.

True. *Hint:* Consider the Mayer–Vietoris long exact sequence on reduced homology associated to the decomposition $X = U \cup V$.

(h) Let d be the local degree of a map $f : S^n \to S^n$ at a point $x \in S^n$. Then d must be ± 1.

False. For example, view S^2 as the Riemann sphere $S^2 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and consider the map

$$\begin{aligned} f : \hat{\mathbb{C}} &\longrightarrow \hat{\mathbb{C}} \\ z &\longmapsto \begin{cases} z^3, & z \in \mathbb{C} \\ \infty, & z = \infty \end{cases} \end{aligned}$$

We proved on Homework 11 #2 that the local degree of this map at $z = 0$ is 3.

(i) Let M be a closed, smoothly embedded submanifold of $\mathbb{R}^n$, and let $\mathbb{R}^n/M$ be the quotient collapsing M to a point. Then $\widetilde{H}_k(\mathbb{R}^n/M) \cong \widetilde{H}_{k-1}(M)$ in each degree k.

True. *Hint:* The tubular neighbourhood theorem implies that $(\mathbb{R}^n, M)$ is a good pair, so the result follows from the long exact sequence of a pair.

(j) There is no path-connected space X with universal cover $\tilde{X}$ satisfying $H_1(\tilde{X}) = \mathbb{Z}^2$.

True. *Hint:* The fundamental group of the universal cover is trivial, and so its abelianization $H_1(\tilde{X})$ is trivial.

(k) There is no path-connected space X with universal cover $\tilde{X}$ satisfying $H_2(\tilde{X}) = \mathbb{Z}^2$.

False. For example, the space $S^2 \vee S^2$ is simply connected and is therefore its own universal cover. It has homology $H_2(S^2 \vee S^2) \cong H_2(S^2) \oplus H_2(S^2) \cong \mathbb{Z}^2$.

(l) Let X be a simply connected, finite CW complex. If $f : X \to X$ is homotopic to the identity, it must have a fixed point.

False. This would be true for a finite CW complex with nonzero Euler characteristic, but the condition of being simply connected is not sufficient. For example, S^3 is simply connected and has a finite CW complex structure. Homework 11 #1 implies that the antipodal map $S^3 \to S^3$ (or on any odd-dimensional sphere) is homotopic to the identity map.