Midterm Exam I

Math 592
18 February 2021
Jenny Wilson

Name: ____________________

Instructions: This exam has 2 questions for a total of 20 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

You have 60 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	9	
2	11	
Total:	20	

Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

1. (9 points) For each of the following statements: if the statement is true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) If two open sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are homotopy equivalent, then $n = m$.

   (b) Let X be a space. Suppose we considered homotopies of paths in X **not** rel $\{0, 1\}$. Then every path would be nullhomotopic.

   (c) The quotient of a CW complex X by any subspace A (not necessarily a subcomplex) has a natural CW complex structure.

(d) Let $\gamma : I \to X$ be a loop in a CW complex X. Then the image of γ is contained in a finite subcomplex of X.

(e) Let $\mathscr{C}$ be a category, and let $f : X \to Y$ be a monomorphism in $\mathscr{C}$. Then the image of f under any covariant functor will be a monomorphism.

(f) There does not exist a retraction from the torus $\Sigma_1 = S^1 \times S^1$ to its subspace

$$A = \Big(S^1 \times \{(1,0)\}\Big) \cup \Big(\{(0,1)\} \times S^1\Big) \cong S^1 \vee S^1.$$

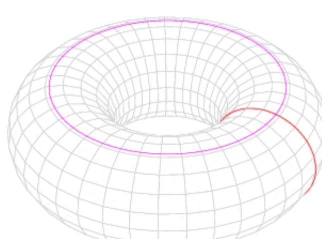


Figure 1: The subspace $A \subseteq \Sigma_1$

(g) If a continuous map of spaces $f : X \to Y$ is surjective, then the induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is surjective.

(h) There exists no covering map from $S^1 \vee S^1$ to S^1.

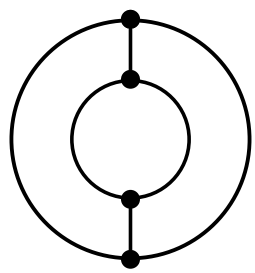
(i) Suppose that a path-connected space X is a union of open, contractible subsets whose pairwise intersections are path-connected. Then $\pi_1(X) = 0$.

2. For each of the following spaces X,

   - determine a presentation for the fundamental group
   - describe loops in the space representing the generators, either by written description or by a picture. (Please describe loops in the space X, not just in a homotopy-equivalent space).

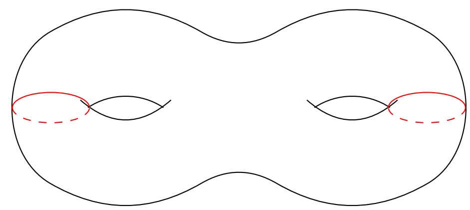
   You do not need to give rigorous proofs, but explain the steps in your reasoning.

   (a) (2 points) X is the graph below.



   (b) (2 points) $X = (S^1 \vee S^2) \times S^\infty$.

(c) (2 points) The space X is constructed by gluing two disks into the surface Σ_2. Each disk is glued in by identifying its boundary homeomorphically with one of the loops in Σ_2 shown below. (Graphics credit: Salman Siddiqi)



(d) (2 points) The space X is constructed from $\mathbb{R}^5$ by deleting a 3-dimensional subspace V through the origin.

(e) (3 points) The space X is the quotient of the polygonal planar shape below by the edge identifications shown. (Shaded regions are part of X, white regions are not).