Midterm Exam II

Math 592
18 March 2021
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Name: ____________________

Instructions: This exam has 4 questions for a total of 15 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 60 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

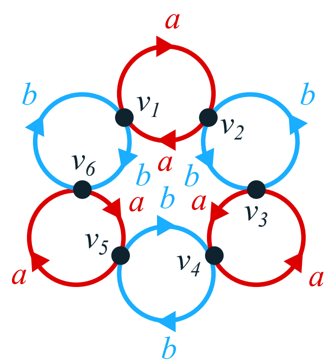
Jenny is available to answer questions.

Question	Points	Score
1	4	
2	4	
3	3	
4	4	
Total:	15	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
  (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

1. (4 points) Identify $\pi_1(S^1 \vee S^1)$ with the free group F_2 on a, b in our conventional way. Consider the following cover of $p : X \to S^1 \vee S^1$. Answer each of the following. **No justification necessary.**



(a) State a free generating set for $H = p_*(\pi_1(X, v_1))$.

One solution: $\langle a^2, b^2, ab^2a^{-1}, (ab)a^2(ab)^{-1}, (aba)b^2(aba)^{-1}, ababab, ababa^{-1}b, \rangle$

(b) Circle one: this cover is . . . `REGULAR` `NOT REGULAR`

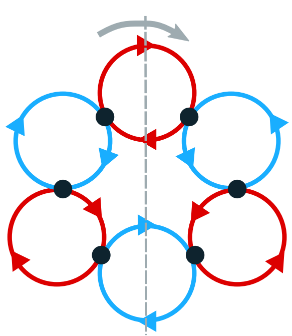
(c) State the deck group for this cover (either by name or by a group presentation).

The deck group is the symmetry group of a regular triangle, equivalently, the symmetric group S_3. Using parts (a) and (b) we can write a presentation for it, (omitting conjugates of relations for efficiency): $F_2/H \cong \langle a, b \mid a^2, b^2, ababab, ababa^{-1}b\rangle$.

(d) Let x_0 be the wedge point in $S^1 \vee S^1$. The elements in the fibre over x_0 are labelled $v_1, v_2, \ldots v_6$. State the action of $a \in F_2$ on the fibre as a permutation of these 6 vertices.

In cycle notation: $(v_1\, v_2)(v_3\, v_4)(v_5\, v_6)$

(e) Is $a \in F_2$ in the normalizer of $H = p_*(\pi_1(X, v_1))$? If so, describe the deck transformation of (X, v_1) defined by a (using the picture below to illustrate). If not, write "`No deck map exists`".



The element $a \in F_2$ is in the normalizer (the cover is regular, so every element is). The deck transformation τ associated to the lift of a starting at v_1 is the unique deck transformation taking v_1 to v_2. This is reflection across the central vertical axis, as shown, while swapping the two red edges and swapping the two blue edges that intersect the vertical axis in order to preserve their edge directions.

2. (4 points) (a) Let D^2 be the closed 2-disk. Suppose G is a group acting on D^2 by a covering space action. Prove that G must be the trivial group.

Solution. A covering space action of G is, by definition, an action by continuous maps $D^2 \to D^2$ such that every $x \in D^2$ has a neighbourhood U_x with $g(U_x)$ disjoint from U_x whenever g is not the identity. In particular, a non-identity element g cannot fix any point $x \in D^2$. However, we proved that every continuous map $D^2 \to D^2$ has a fixed point. We conclude that G is the trivial group.

(b) Let X be a path-connected, locally path-connected space. Prove that any cover $D^2 \to X$ must be a homeomorphism.

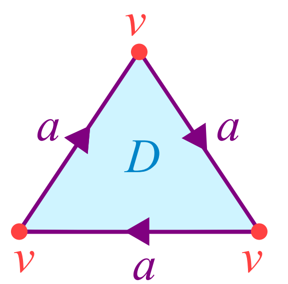
Solution. Suppose $p : (D^2, \tilde{x_0}) \to (X, x_0)$ is a covering map. Since X is path-connected and locally path-connected, our proposition on the structure of deck transformation applies: if $H = p_*(\pi_1(D^2, \tilde{x_0}))$ is normal in $\pi_1(X, x_0)$, then the cover is regular and the Deck group is isomorphic to $\pi_1(X, x_0)/H$.

The trivial subgroup must be normal in $\pi_1(X, x_0)$, and so the cover $p : D^2 \to X$ is a regular cover with deck group isomorphic to $\pi_1(X, x_0)$. But we proved that the deck group acts by a covering space action, so by part (a), $\pi_1(X, x_0)$ is the trivial group.

Then H is index 1 in $\pi_1(X, x_0)$, and p is a 1-sheeted cover. A cover is a local homeomorphism, thus is an open map. A 1-sheeted cover is by definition a cover where every element x in the codomain has exactly one preimage, equivalently, it is a bijection. The map p is open and bijective, therefore is a homeomorphism.

In contrast, the open disk $\mathring{D}^2$ covers lots of spaces – including the torus and the surface Σ_g of genus g for all $g \geq 1$.

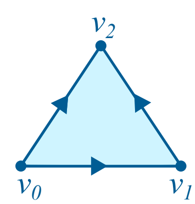
3. (3 points) Compute the simplicial homology groups $H_n(X)$ of the following Δ-complex X. Describe each homology group in the sense of the structure theorem for finitely generated abelian groups.



Solution. The complex X has a single simplex in dimensions $0, 1, 2$, and so

$$C_0(X) = \mathbb{Z}v \qquad C_1(X) = \mathbb{Z}a \qquad C_2(X) = \mathbb{Z}D.$$

We order the vertices of our 2-simplex $[v_0, v_1, v_2]$ as shown, and observe that $[v_0, v_1]$ is glued to $-a$, $[v_1, v_2]$ is glued to $-a$, and $[v_0, v_2]$ is glued to a. All three vertices are glued to v.



Thus our boundary maps are

$$\begin{aligned} \partial_2 : C_2(X) &\longrightarrow C_1(X) \\ \mathbb{Z}D &\longrightarrow \mathbb{Z}a \\ D &\longmapsto (-a) - (a) - (-a) = -3a \end{aligned} \qquad \begin{aligned} \partial_1 : C_1(X) &\longrightarrow C_2(X) \\ \mathbb{Z}a &\longrightarrow \mathbb{Z}v \\ a &\longmapsto v - v = 0 \end{aligned}$$

Then our chain complex is

$$\longrightarrow 0 \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0$$

$$\ker(\partial_2) = 0 \qquad \ker(\partial_1) = \mathbb{Z}a \qquad \ker(\partial_0) = \mathbb{Z}v$$

$$\text{im}(\partial_3) = 0 \qquad \text{im}(\partial_2) = 3\mathbb{Z}a \qquad \text{im}(\partial_1) = 0$$

We conclude

$$H_2(X) = \frac{0}{0} = 0 \qquad H_1(X) = \frac{\mathbb{Z}a}{3\mathbb{Z}a} \cong \mathbb{Z}/3\mathbb{Z} \qquad H_0(X) = \frac{\mathbb{Z}v}{0} \cong \mathbb{Z}.$$

4. (4 points) For each of the following statements: if the statement is true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Every continuous map from $\mathbb{R}\mathrm{P}^2$ to an n-torus $(S^1)^n$ is nullhomotopic.

   True.

   Hint: $\pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}/2\mathbb{Z}$ is finite, so any group homomorphism to $\pi_1((S^1)^n) = \mathbb{Z}^n$ is trivial. Thus every map $\mathbb{R}\mathrm{P}^2 \to (S^1)^n$ lifts to the (contractible) universal cover $\mathbb{R}^n$ of $(S^1)^n$, and is therefore is nullhomotopic.

   (b) Let X be a path-connected, locally path-connected space. Then X has a universal cover.

   False.

   X must also be semi-locally simply-connected. For example, the Hawaiian earring is path-connected and locally path-connected but not semi-locally simply connected, and so cannot have a universal cover by Homework 5 # 3(a).

   (c) Let $\tilde{X} \to X$ be a (not necessarily connected) covering space, and let τ be a deck map $\tilde{X} \to \tilde{X}$. If τ fixes a point, then τ is the identity.

   False.

   Let X be a nonempty space, let $X_1 = X_2 = X_3 = X$. Then we can define a cover $X_1 \sqcup X_2 \sqcup X_3 \to X$ that restricts to the identity map on each X_i. Define $\tau : X_1 \sqcup X_2 \sqcup X_3 \to X_1 \sqcup X_2 \sqcup X_3$ to map X_1 by the identity to X_2, map X_2 by the identity to X_1, and fix X_3 pointwise. Then τ is a nontrivial deck map that fixes X_3 pointwise.

   (d) Let F_2 be the free group on 2 letters a, b. Then any finitely-generated nontrivial subgroup of F_2 has finite index.

   False.

   Let $H = \langle a \rangle$. Then H is finitely generated, but the associated cover is infinite-sheeted, so H is infinite index.