Midterm Exam I

Math 592
18 February 2021
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Name: ______________________________

Instructions: This exam has 2 questions for a total of 20 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

You have 60 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	9	
2	11	
Total:	20	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n\geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

1. (9 points) For each of the following statements: if the statement is true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) If two open sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are homotopy equivalent, then $n = m$.

   False. For example, take $A = \mathbb{R}^n$ and $B = \mathbb{R}^m$ for any n, m. We proved that both are homotopy equivalent, since both are contractible.

   (b) Let X be a space. Suppose we considered homotopies of paths in X **not** rel $\{0, 1\}$. Then every path would be nullhomotopic.

   True.

   Hint: This is a consequence of the contractibility of the interval I. For a path γ, consider the homotopy $\gamma_t(s) = \gamma((1 - t)s)$.

   (c) The quotient of a CW complex X by any subspace A (not necessarily a subcomplex) has a natural CW complex structure.

   False. For example, consider the construction from Homework 3 # 4(g). Let X be the circle; it is a CW complex with 1 vertex and 1 edge. Let A be the complement of a point in X. Then the topology on the two-point set X/A is not discrete (in particular, not Hausdorff), so cannot admit a CW complex structure by Homework 1 #5. In general, any non-Hausdorff quotient provides a counterexample.

(d) Let $\gamma : I \to X$ be a loop in a CW complex X. Then the image of γ is contained in a finite subcomplex of X.

True.

Hint: Since I is compact, the image of γ is compact, and therefore contained in a finite union of cells by Homework 1 #6(b). Use Homework 1 #6 to argue that a finite union of cells is contained in a finite subcomplex.

(e) Let $\mathscr{C}$ be a category, and let $f : X \to Y$ be a monomorphism in $\mathscr{C}$. Then the image of f under any covariant functor will be a monomorphism.

False. For example, consider the category $\mathscr{C}$ with only two objects $\blacksquare$ and $\bigstar$ and only one non-identity morphism $f : \blacksquare \to \bigstar$. $\mathscr{C}$ is shown below.

$$\mathrm{id}_{\blacksquare} \circlearrowright \blacksquare \xrightarrow{f} \bigstar \circlearrowleft \mathrm{id}_{\bigstar}$$

The morphism f vacuously satisfies the monomorphism condition. Any choice of morphism $g : A \to B$ in a category $\mathscr{D}$ determines a functor $\mathscr{C} \to \mathscr{D}$ mapping f to g. So we can choose g to be any non-monic morphism, say, a non-injective function in the category of sets.

(f) There does not exist a retraction from the torus $\Sigma_1 = S^1 \times S^1$ to its subspace

$$A = \Big(S^1 \times \{(1,0)\}\Big) \cup \Big(\{(0,1)\} \times S^1\Big) \cong S^1 \vee S^1.$$

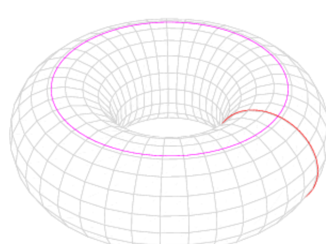


Figure 1: The subspace $A \subseteq \Sigma_1$

True.

Hint: By Homework 3 #1(a), a retraction $S^1 \times S^1$ to $A \cong S^1 \vee S^1$ would induce a surjection from $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ to $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$. But no such surjective map can exist, since $\mathbb{Z}^2$ is abelian and the free group $\mathbb{Z} * \mathbb{Z}$ is not.

(g) If a continuous map of spaces $f : X \to Y$ is surjective, then the induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is surjective.

False. For example, the quotient of a closed interval by its boundary defines a surjective map $I \to S^1$. But $\pi_1(I) \cong 0$ and $\pi_1(S^1) \cong \mathbb{Z}$, so the induced map $0 \to \mathbb{Z}$ cannot be surjective.

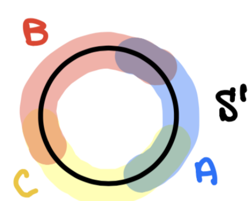
(h) There exists no covering map from $S^1 \vee S^1$ to S^1.

True.

Hint: Verify that no open subset of S^1 is homeomorphic to any neighbourhood of the wedge point in $S^1 \vee S^1$.

(i) Suppose that a path-connected space X is a union of open, contractible subsets whose pairwise intersections are path-connected. Then $\pi_1(X) = 0$.

False. We must also assume that the intersection of the open subsets is nonempty, so we can choose a common basepoint. For example, the space S^1 has nontrivial fundamental group $\mathbb{Z}$, but can be decomposed into a union of open contractible sets A, B, C with contractible pairwise intersections, as shown below.



2. For each of the following spaces X,

- determine a presentation for the fundamental group
- describe loops in the space representing the generators, either by written description or by a picture. (Please describe loops in the space X, not just in a homotopy-equivalent space).

You do not need to give rigorous proofs, but explain the steps in your reasoning.

(a) (2 points) X is the graph below.

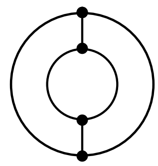
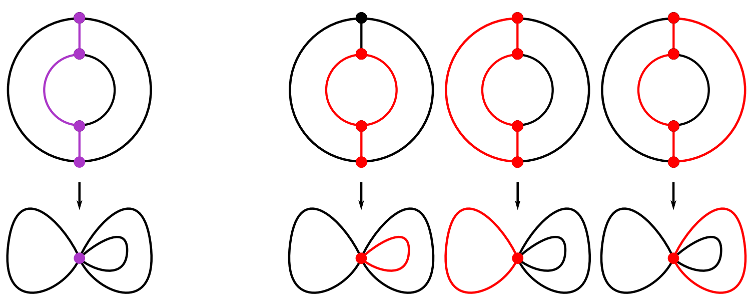


Figure 2: The graph X

We can construct a homotopy equivalence between X and a wedge of three circles by taking the quotient by a choice of maximal tree T. We obtain free generators for π_1 by taking a lift of each circle in X/T. So $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \cong \langle a, b, c \mid \rangle$, with one choice of generating set shown in the figure below, with loops based at the bottommost vertex of X.



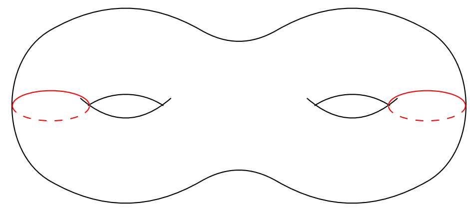
(b) (2 points) $X = (S^1 \vee S^2) \times S^\infty$.

We proved: $\pi_1(S_1) \cong \mathbb{Z}$ $\quad$ $\pi_1(S^2) \cong 0$ $\quad$ $\pi_1(S^\infty) \cong 0$ [since $S^\infty \simeq *$].

$$\begin{aligned}\text{Thus, } \pi_1\Big((S^1 \vee S^2) \times S^\infty\Big) &\cong \pi_1(S^1 \vee S^2) \times \pi_1(S^\infty) \\ &\cong \Big(\pi_1(S^1) * \pi_1(S^2)\Big) \times \pi_1(S^\infty) \\ &\cong (\mathbb{Z} * 0) \times 0 \\ &\cong \mathbb{Z}.\end{aligned}$$

$\pi_1(X)$ has presentation $\langle a \mid \rangle$, where a is represented by a loop $t \mapsto (\omega(t), x_0)$, with $x_0 \in S^\infty$, and $\omega(t)$ is obtained by composing the loop $t \mapsto (\cos(t), \sin(t))$ in S^1 with the inclusion $S^1 \hookrightarrow S^1 \vee S^2$.

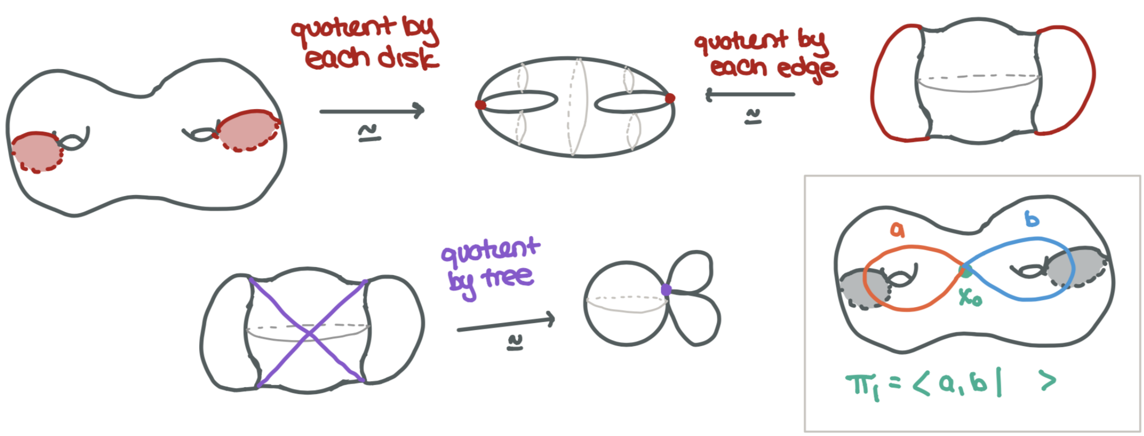
(c) (2 points) The space X is constructed by gluing two disks into the surface Σ_2. Each disk is glued in by identifying its boundary homeomorphically with one of the loops in Σ_2 shown below. (Graphics credit: Salman Siddiqi)



Repeatedly apply our result that the quotient of a CW complex by a contractible subcomplex is a homotopy equivalence. We determine that X is homotopy equivalent to $S^1 \vee S^1 \vee S^2$. Hence

$$\pi_1(X) \;\cong\; \pi_1(S^1) * \pi_1(S^1) * \pi_1(S^2) \;\cong\; \mathbb{Z} * \mathbb{Z} * 0 \;\cong\; \langle a, b \mid \; \rangle,$$

and we can determine generators by tracing images/lifts the two circles S^1 through the sequence of equivalences.



(d) (2 points) The space X is constructed from $\mathbb{R}^5$ by deleting a 3-dimensional subspace V through the origin.

Choose a basis b_1, b_2, b_3, b_4, b_5 of $\mathbb{R}^5$ so that $V = \operatorname{span}(b_3, b_4, b_5)$, ie, in coordinates,

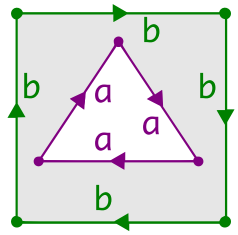
$$V = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 = x_2 = 0\}.$$

Then

$$\begin{aligned} X &= \{(x_1, x_2, x_3, x_4, x_5) \mid x_1, x_2 \text{ not both zero}\} \\ &= \{(x_1, x_2) \mid x_1, x_2 \text{ not both zero}\} \times \{(x_3, x_4, x_5)\} \\ &= (\mathbb{R}^2 - 0) \times \mathbb{R}^3 \end{aligned}$$

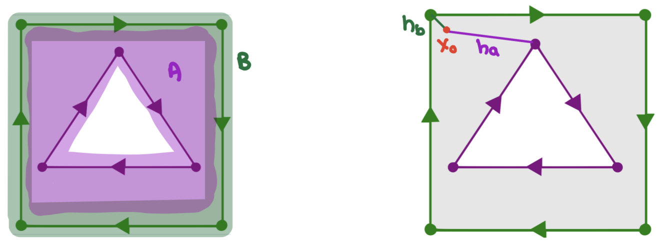
Thus $\pi_1(X) = \pi_1(\mathbb{R}^2 - 0) \times \pi_1(\mathbb{R}^3) = \mathbb{Z} \times 0 = \mathbb{Z}$, and has presentation $\langle a \mid \; \rangle$, where a is (eg) represented by the loop $t \mapsto (\cos(t), \sin(t), 0, 0, 0)$.

(e) (3 points) The space X is the quotient of the polygonal planar shape below by the edge identifications shown. (Shaded regions are part of X, white regions are not).



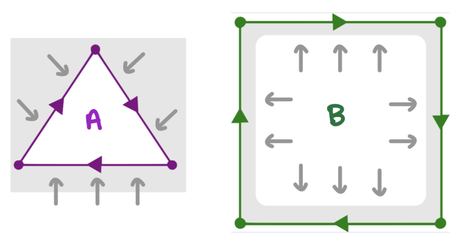
By abuse of notation, we write a (and similarly b) to mean both the subspace of X, and the loop $I \to S^1 \hookrightarrow X$ defined by the inclusion $S^1 \hookrightarrow X$ of the subspace a.

One approach to this problem is to use the Van Kampen theorem. Consider the decomposition of X as the union of open subsets A and B shown. Observe that A and B are path-connected, and have non-empty, path-connected intersection.



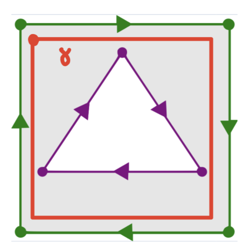
We choose a basepoint x_0 in $A \cap B$, and paths h_a, h_b from x_0 to points on a and b, respectively, as shown above. Call $\tilde{a} = [h_a \cdot a \cdot \overline{h_a}]$ and $\tilde{b} = [h_b \cdot b \cdot \overline{h_b}]$.

The set A deformation retracts onto the circle a by a straight-line homotopy F_t sketched opposite. The map $(F_1)_*$ maps $\tilde{a}$ to the class of the loop a. We conclude that $\pi_1(A, x_0) \cong \mathbb{Z}$ and is generated by $\tilde{a}$. Similarly B deformation retracts onto b, and its fundamental group $\mathbb{Z}$ is generated by $\tilde{b}$.



The intersection $A \cap B$ is homotopic to a circle S^1, and its fundamental group $\mathbb{Z}$ is generated by the embedded loop γ shown. The loop γ is path homotopic to $(h_a \cdot a \cdot \overline{h_a})^3$ and to $(h_b \cdot b \cdot \overline{h_b})^4$. Hence we obtain the presentation

$$\langle \tilde{a}, \tilde{b} \mid \tilde{a}^3 = \tilde{b}^4 \rangle.$$



 End of exam.