## Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit $n$-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ ($n$-sphere) (we sometimes view $S^1$ as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- $\Sigma_g$ closed genus-$g$ surface
- $\mathbb{R}\mathrm{P}^n$ real projective $n$-space
- $\mathbb{C}\mathrm{P}^n$ real complex $n$-space

## Practice problems

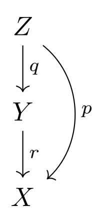
1. Let $X$ be a Hausdorff space. Prove that any cover $\tilde{X}$ of $X$ must be Hausdorrff.
   *Remark:* The converse is not true!

2. Let $p : \tilde{X} \to X$ be a covering map with $p^{-1}(x)$ finite and nonempty for all $x \in X$ . Show that $\tilde{X}$ is compact Hausdorff if and only if $X$ is compact Hausdorff.

3. Let $p : \tilde{X} \to X$ be a covering space. Prove that $X$ has a basis of open subsets that are evenly covered by $p$.

4. For each of the following maps: prove that the map is a covering map, or prove that it is not a covering map.
   (a) The defining quotient map $S^n \to \mathbb{R}\mathrm{P}^n$.
   (b) The defining quotient map $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$.

5. Let $X, Y, Z$ be path-connected, locally path-connected, and semi-locally simply-connected spaces. Let $p, q, r$ be continuous maps with $p = r \circ q$. Show that, if $p$ and $q$ are covering maps, then so is $r$.




6. Let $p : \tilde{X} \to X$ be a covering space, and let $\{U_\alpha\}$ be an open cover of $X$ with the property that each open subset $U_\alpha$ is evenly covered by $p$. Let $f : I \to X$ be a continuous path. Prove that there is a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each $i$, $f|_{[t_i, t_{i+1}]} \subseteq U_i$ for some $U_i \in \{U_\alpha\}$.

7. (a) Suppose that $X$ is a graph. Show that, if $X$ is simply connected, then $X$ is contractible.
   (b) Let $X$ be a graph. Conclude that the universal cover of $X$ is contractible.
   (c) Let $X$ be a graph, and let $L$ be a connected space with a finite fundamental group. Prove that every continuous map from $L$ to $X$ is nullhomotopic.

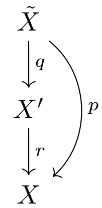
8. Let $p : \tilde{Y} \to Y$ be a covering map, and let $f : X \to Y$ be any map.
   (a) Define a *lift* of $f$ to $\tilde{Y}$, and give sufficient conditions for a lift $\tilde{f}$ to exist.
   (b) Explain how we constructed the lift $\tilde{f}$.
   (c) In what sense is the lift unique? Show by example how, when $X$ is not connected, the lift $\tilde{f}$ need not satisfy our uniqueness condition.

9. Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering map. Prove that the induced map $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ is injective.

10. Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering map, and let $f : (S_n, s_0) \to (X, x_0)$ be a based map. Prove that $f$ lifts to a unique based map $f : (S_n, s_0) \to (\tilde{X}, \tilde{x_0})$.

11. Let $n \geq 2$. Let $f : (S^n, s_0) \to (S^1 \times S^1, x_0)$ be a map of based spaces. Show that $f$ is nullhomotopic rel $s_0$. *Remark:* We can use this result to prove that $\pi_n(S^1 \times S^1, x_0) = 0$ for $n \geq 2$.

12. Let $X$ be a path-connected, locally path-connected, semi-locally simply connected space.
    (a) Show that the universal cover $p : \tilde{X} \to X$ of $X$ is universal in the following sense: Given any covering map $r : X' \to X$, there is covering map $q : \tilde{X} \to X'$ making the following diagram commute.




    (b) Is the covering map $q : \tilde{X} \to X'$ unique?

13. (a) Let $p : \tilde{X} \to X$ be a covering map. Explain why, if $\tilde{X}$ is simply connected, then it must be the universal cover (or, more precisely, it must be isomorphic as a cover to the universal cover).
    (b) Read Tai-Danae Bradley's 'recipe' for constructing the universal cover of a wedge sum, and describe the procedure in your own words.
    `https://www.math3ma.com/blog/a-recipe-for-the-universal-cover-of-x-y`
    (c) Give explicit descriptions (possibly by picture) of the universal covers of the following spaces.

    (i) $S^1 \times S^1 \times S^1$
    (ii) $S^1 \vee S^1 \vee S^1$
    (iii) $S^2 \vee S^1$
    (iv) $\mathbb{R}\mathrm{P}^n$, $n \geq 2$
    (v) $\mathbb{C}\mathrm{P}^n$

14. Suppose that $X$ is not semi-locally simply connected. Explain why $X$ cannot have a universal cover.

15. The *Euler number* of a finite graph $X$ is the number of vertices of $X$ minus the number of edges of $X$.
    (a) Suppose $X$ is a finite connected graph with Euler number $\chi(X)$. What is the rank of the free group $\pi_1(X)$?
    (b) If $X$ is a finite graph and $\tilde{X} \to X$ is an $n$-sheeted cover of $X$, what is the relationship between the Euler number of $X$ and the Euler number of $\tilde{X}$?

16. Describe all (based) isomorphism classes of regular 3-sheeted covers of $S^1 \vee S^1$.

17. (a) **(Centralizer).** Let $G$ be a group. Let $S$ be a subgroup of $G$ (or, more generally, a subset). The *centralizer* of $S$ is defined to be

$$C_G(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\}.$$

    Prove that $C_G(S)$ is a subgroup of $G$.
    (b) Explain the difference between the normalizer $N_G(S)$ of $S$ and the centralizer $C_G(S)$ of $S$.
    (c) Prove that $C_G(S)$ is contained in $N_G(S)$, and that it is a normal subgroup.
    (d) Under what conditions on $S$ will we have containment $S \subseteq C_G(S)$?
    (e) We have another name for the subgroup $C_G(G)$. What is it?

18. (a) Consider the transposition (12) in the symmetric group $S_4$. What is the normalizer of the subgroup $\langle(12)\rangle$ in $S_4$?

   (b) What is the normalizer of the subgroup $\langle a \rangle$ in the free group $F_2$ on $a, b$ ?

19. **(Topology QR Exam, Jan 2021).** Let $X = \mathbb{R}P^3$ and let $Y = S^1 \vee S^1$.
   (a) Are all maps $f : X \to Y$ null-homotopic?
   (b) Are all maps $f : Y \to X$ null-homotopic?

   For each of the above, give a proof if the answer is "yes" and give an example if the answer is "no".

20. **(Topology QR Exam, Aug 2020).** For each of the following cases, determine if there exists a covering space $f : X \to Y$. (If yes, then construct it; if not, then give a proof).
   (a) $X$ is homotopy equivalent to $S^1 \times S^1$ and $Y$ is homotopy equivalent to $S^1 \vee S^1$.
   (b) $X$ is homotopy equivalent to $S^1 \vee S^1$ and $Y$ is homotopy equivalent to $S^1 \times S^1$.

21. **(Topology QR Exam, May 2018).** Consider the 1-point compactification $X = \mathbb{R}^3 \cup \{\infty\}$ of $\mathbb{R}^3$. Now let $\mathbb{Z}/2$ act on $X$ where the generator sends $x \mapsto -x$ for $x \in \mathbb{R}^3$, and $\infty \mapsto \infty$. Let $Y$ be the orbit space of $X$ with the quotient topology. How many non-isomorphic connected covering spaces (in the unbased sense) does $Y$ have? Prove your answer.

22. **(Topology QR Exam, Jan 2018).** Let $X$ be a graph with one vertex and two edges. Does there exist a connected covering $f : Y \to X$ which is regular and a connected covering $g : Z \to Y$ which is regular such that $fg : Z \to X$ is not a regular covering? Prove your answer.

23. **(Topology QR Exam, May 2017).** Let $X$ be a connected CW-complex whose fundamental group is $\Sigma_3$, the group of all permutations on 3 elements.
   (a) How many isomorphism classes of objects are there in the category $\mathrm{Cov}_0(X)$ of connected covering spaces of $X$ and continuous maps commuting with the covering map?
   (b) How many isomorphism classes of objects of $\mathrm{Cov}_0(X)$ have degree 2?
   (c) How many isomorphism classes of objects of $\mathrm{Cov}_0(X)$ are regular coverings?

24. **Definition (Abelian cover).** Let $X$ be path-connected, locally path-connected, and semi-locally simply-connected. A cover $p\tilde{X} \to X$ is *abelian* if it is a regular cover with an abelian deck group.
   (a) Prove that $X$ has an abelian cover $U$ that is universal in the sense that it is a cover of every other abelian cover of $X$.
   (b) Verify that the cover $U \to X$ is unique up to isomorphism of covers.
   (c) What is $U$ when $X = S^1 \vee S^1$?

25. Let $\Sigma_g$ be a genus-$g$ surface, and let $n \leq 2g$. Prove or disprove: $\Sigma_g$ has a regular covering space with deck group $\mathbb{Z}^{2g}$.

26. Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a path-connected covering map, and let $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$. Prove that, if $[\gamma] \in \pi_1(X, x_0)$, then there is a point $\tilde{x_1} \in p^{-1}(x_0)$ with $p_*(\pi_1(\tilde{X}, \tilde{x_1})) = [\gamma]^{-1} H [\gamma]$.

27. Find free generating sets for the kernels of the following homomorphisms.
   (a)
$$\begin{aligned} h : F_2 &\longrightarrow \mathbb{Z} \\ a &\longmapsto 1 \\ b &\longmapsto 0 \end{aligned}$$
   (b)
$$\begin{aligned} h : F_2 &\longrightarrow \mathbb{Z} \\ a &\longmapsto 1 \\ b &\longmapsto 1 \end{aligned}$$

(c)

$$h : F_2 \longrightarrow \mathbb{Z}$$
$$a \longmapsto 1$$
$$b \longmapsto 2$$

28. **(Topology QR Exam, May 2020).** Describe a set of free generators of the subgroup of the free group on two generators $a, b$ generated by b and all the conjugates of $a^2$, $b^2$, and $(ab)^3$. Is this a normal subgroup?

29. **(Topology QR Exam, Jan 2020).** Describe a set of free generators of the subgroup of the subgroup of the free group on two generators $a, b$ generated by all conjugates of $aba^{-1}b^{-1}$.

30. **(Topology QR Exam, May 2019).** Let $F$ be a free group on two generators $a, b$ and let $h : F \to \mathbb{Z}/2 \times \mathbb{Z}/2$ be an onto homomorphism. Is $\text{Ker}(h)$ a free group? If so, find its free generators.

31. Build an $n$-sheeted cover of $S^1 \vee S^1$, with vertices labelled $1, 2, \ldots, n$, such that the action of $a$ on the vertices is given by the permutation (in cycle notation):
    (a) (123)
    (b) (12)(3)
    (c) (1)(2)(3)
    (d) (12)(34)

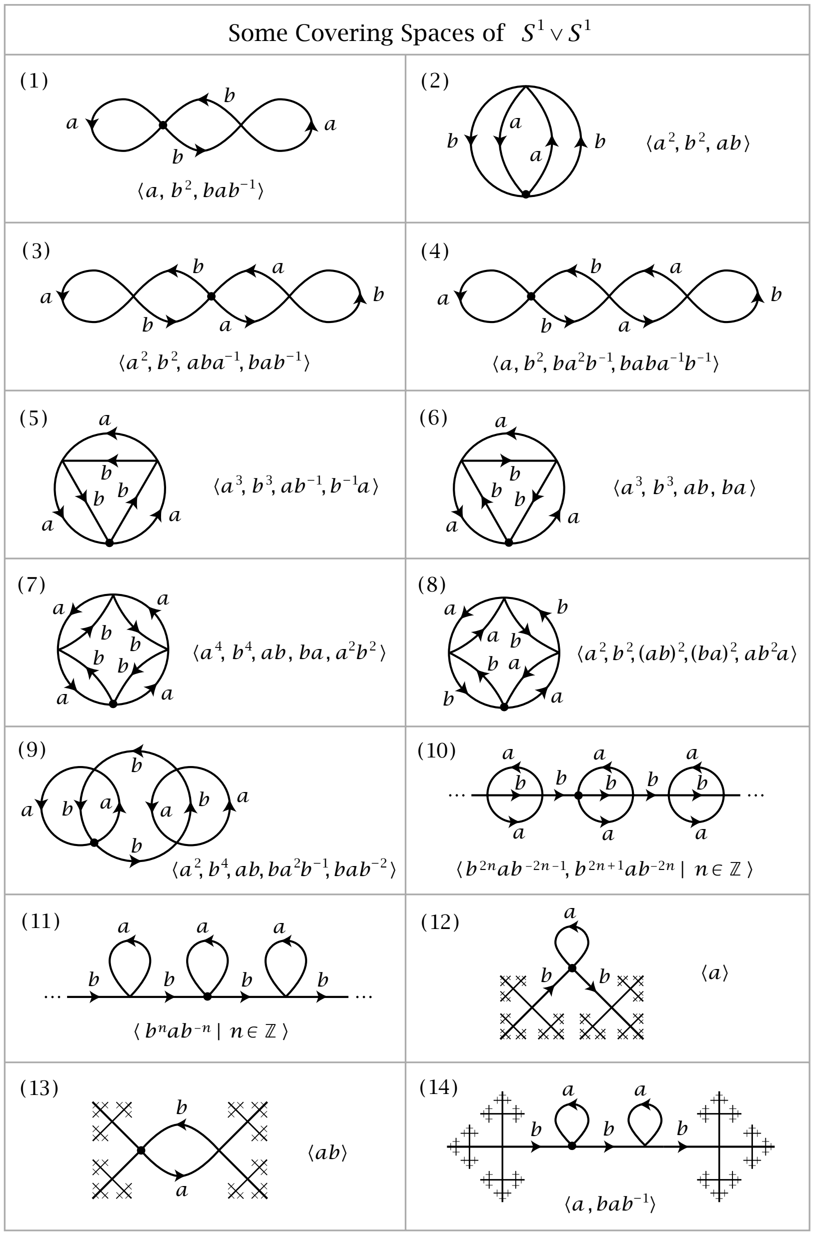
32. Let $p : \tilde{X} \to X$ be a connected, regular covering map, and let $x_0 \in X$. Let $G(\tilde{X})$ be the deck group. Build a bijection of sets between $G(\tilde{X})$ and the fibre $p^{-1}(x_0)$.

33. Suppose that $p : \tilde{X} \to X$ is a path-connected cover. Recall that we defined the cover to be *regular* if, for any $x \in X$, the group of Deck transformations of the cover acts transitively on the fibre $p^{-1}(x)$. Suppose that there is a single point $x_0 \in X$ such that the group of Deck transformations of the cover acts transitively on the fibre $p^{-1}(x_0)$. Prove that the cover is regular.

34. Let $(X, x_0)$ be a based space, and let $p : \tilde{X} \to X$ be a cover.
    (a) Explain why the subgroup $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is independent of the choice of basepoint $\tilde{x_0} \in p^{-1}(x_0)$ if and only if $H$ is normal.
    (b) Let $\gamma \in \pi_1(X, x_0)$. Show by example that the deck transformation of $\tilde{X}$ defined by $\gamma$ may depend on the choice of basepoint $\tilde{x_0}$, even if $H$ is normal.

    (Note that the subgroup $H$, its normalizer $N(H)$, and the action of an element of $N(H)$ all depend on our choice of basepoint $\tilde{x_0}$).

35. Consider Hatcher's covers $\tilde{X}$ of $S^1 \vee S^1$ in the table below. Each cover has a distinguished basepoint $\tilde{x_0}$ (marked by a black dot) in the preimage of the single vertex $x_0$ in $S^1 \vee S^1$. We identify $\pi_1(S^1 \vee S^1, x_0)$ with tthe free group $F_2$ on letters $a, b$. For each cover, answer the following questions.
    (a) How many sheets is the cover?
    (b) What is $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ as a subgroup of $\pi_1(S^1 \vee S^1, x_0) = F_2$? Give a free generating set.
    (c) Is the cover $\tilde{X}$ regular?
    (d) Describe the group of deck transformations of $\tilde{X}$.
    (e) Number the vertices of $\tilde{X}$, and compute the permutations on the vertices defined by the actions of $a, b$, and $ab$ in $\pi_1(S^1 \vee S^1, x_0)$.
    (f) In our proof identifying the deck group of $\tilde{X}$ with $N(H)/H$, we defined an action of $N(H) \subseteq F_2$ by deck transformations. For each of the covers, for the elements $\gamma = a$ and $\gamma = b$ in $F_2$, determine whether $\gamma$ is in $N(H)$, and, if so, describe the associated deck transformation of $\tilde{X}$.




36. Describe the universal cover of $S^1 \vee S^1$. Choose a cover from Hatcher's table, and explain/illustrate how it can be constructed as a quotient of the universal cover by a suitable choice of covering action.

37. Let $L(p,q)$ be the lens space with parameters $p, q$ from Homework 7. Describe all (unbased) isomorphism classes of covering spaces of $L(p,q)$, and the system of all intermediate covering maps.

38. Let $p : \tilde{X} \to X$ be a connected cover, and let $x_0 \in X$.
   (a) Let $\tilde{x_1}, \tilde{x_2} \in p^{-1}(x_0)$. Prove that, if a deck transformation mapping $\tilde{x_1}$ to $\tilde{x_2}$ exists, then it is unique.
   (b) Suppose a deck transformation $\tau$ mapping $\tilde{x_1}$ to $\tilde{x_2}$ exists. Explain how to determine where $\tau$ maps an arbitrary point $\tilde{x} \in \tilde{X}$. *Hint:* Recall our construction from our proof of the existence of lifts.

39. Given a group $G$ and a normal subgroup $N$ , show that there exists a normal covering space $\tilde{X} \to X$ with $\pi_1(X) \cong G$, $\pi_1(\tilde{X}) \cong N$, and deck transformation group $G(\tilde{X}) \cong G/N$ .

40. Let $S^{2k-1} \subseteq \mathbb{C}^k \cong \mathbb{R}^{2k}$ be the unit sphere. Define an action of $\mathbb{Z}/m\mathbb{Z}$ on $S^{2k-1}$ by rotation, generated by the map $v \mapsto e^{2\pi i/m}v$. Compute the fundamental group of its orbit space.

41. Let $G_1$ act on $X_1$ and $G_2$ act on $X_2$ by covering space actions.
    (a) Define an action of $G_1 \times G_2$ on $X_1 \times X_1$ by
    $$(g_1, g_2) \cdot (x_1, x_2) = (g_1 \cdot x_1, g_2 \cdot x_2).$$
    Prove this is a covering space action.
    (b) Prove that $(X_1 \times X_2)/(G_1 \times G_2)$ is homeomorphic to $X_1/G_1 \times X_2/G_2$ .

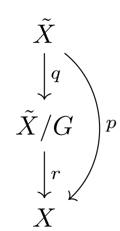
42. Consider the following actions of a group $G$ on a space $X$. Determine which actions are covering space actions.
    (a) $X = S^1$, $G = \mathbb{Z}/2\mathbb{Z}$, and the generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by 180° rotation.
    (b) $X = S^2$, $G = \mathbb{Z}/2\mathbb{Z}$, and the generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by 180° rotation around the vertical axis.
    (c) $X = \mathbb{R}^n$, $G = \mathbb{R}$, and $r \in \mathbb{R}$ acts by
    $$r \cdot (x_1, x_2, x_3 \dots, x_n) = (x_1 + r, x_2, x_3, \dots, x_n).$$
    (d) $X = \mathbb{R}^n$, $G = \mathbb{Z}$, and $z \in \mathbb{Z}$ acts by
    $$z \cdot (x_1, x_2, x_3 \dots, x_n) = (x_1 + z, x_2, x_3, \dots, x_n).$$
    (e) $X = Y^n$ for some space $Y$, $G = S_n$, and $\sigma \in S_n$ acts by
    $$\sigma \cdot (y_1, y_2, \dots, y_n) = (y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)}).$$

43. For each of the following actions: prove that they are covering actions, and identify the quotient (they are spaces we know by name!)
    (a) The group $\mathbb{Z}/2\mathbb{Z}$ acts on $S^1 \times I$. The generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by $(x, t) \mapsto (-x, t)$.
    (b) The group $\mathbb{Z}/2\mathbb{Z}$ acts on $S^1 \times I$. The generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by $(x, t) \mapsto (-x, 1 - t)$.

44. Find a covering space action of $\mathbb{Z}/2\mathbb{Z}$ on the torus so that the quotient is a Klein bottle.

45. Let $G$ be a group with a covering space action on a path-connected, locally path-connected space $\tilde{X}$.
    (a) What can you say about the relationship between $G$ and $\pi_1(X/G)$? (Note: we did not assume $X$ is simply connected).
    (b) For a subgroup $H \subseteq G$, show that $X/H$ is a cover of $X/G$.
    (c) Show that the cover $X/H \to X/G$ is normal if and only if $H$ is a normal subgroup of $G$.
    (d) For subgroups $H_1, H_2$ of $G$, show that the covering space $X/H_1$ and $X/H_2$ of $X/G$ are isomorphic if and only if $H_1$ and $H_2$ are conjugate subgroups of $G$.

46. Let $\tilde{X} \to X$ be a (not necessarily regular) cover, and let $G$ be its group of deck transformations. Let $q : \tilde{X} \to \tilde{X}/G$ be the quotient map to the orbit space $X/G$. Show that there exists a map $r$ making the following diagram commute.



47. Let $\{(C^i_*, d^i_*)\}_{i \in I}$ be a family of chain complexes.
   (a) How should we define the complex $\left(\bigoplus_i C^i_*, d_*\right)$?
   (b) Prove that (for a suitable solution to part (a)),
$$H_n\left(\bigoplus_i C^i_*\right) = \bigoplus_i H_n(C^i_*).$$

48. (a) Let $\{A_n\}_{n \in \mathbb{Z}_{\geq 0}}$ be a family of abelian groups. Construct a chain complex $\{(C_*, d_*)\}$ such that $H_n(C_*) = A_n$.
   (b) Let $\{A_n\}_{n \in \mathbb{Z}_{\geq 0}}$ be a family of finitely generated abelian groups. Construct a chain complex $\{(C_*, d_*)\}$ such that $C_n$ is free abelian for all $n$, and $H_n(C_*) = A_n$.

49. Consider the abelian groups $A$ and subgroups $B$ given below. Compute the isomorphism type of the quotient $A/B$ (in the sense of the structure theorem for finitely generated abelian groups).
   (a) $A = \mathbb{Z}^2$, $B$ is the subgroup generated by $(2, 3)$.
   (b) $A = \mathbb{Z}^2$, $B$ is the subgroup generated by $(2, 4)$.
   (c) $A = \mathbb{Z}^2$, $B$ is the subgroup generated by $(1, 1)$ and $(1, -1)$.
   (d) $A = \mathbb{Z}^2$, $B$ is the subgroup generated by $(2, 1)$ and $(7, 4)$.
   (e) $A = \mathbb{Z}^2$, $B$ is the subgroup generated by $(2, 1)$ and $(1, 3)$.

50. For each of the following spaces, choose a $\Delta$-complex structure, and compute the simplicial homology.
   (a) a selection of finite graphs of your choosing
   (b) a cylinder
   (c) a Klein bottle
   (d) the wedge sum of a closed 2-disk and a circle
   (e) the disjoint union of a cylinder and a circle

51. **True or counterexample.** For each of the following statements: if the statement is true, write "True". If not, state a counterexample. No justification necessary.
*Note:* If the statement is false, you can receive partial credit for writing "False" without a counterexample.
   (a) If $p : \tilde{X} \to X$ is a covering space map, then $p$ cannot be nullhomotopic.
   (b) If $X$ is simply-connected, then $X$ is semi-locally simply-connected.
   (c) If $X$ is connected, then $X$ is semi-locally simply-connected.
   (d) If $X$ is locally simply-connected and semi-locally simply-connected, then $X$ is simply-connected
   (e) Let $p : \tilde{Y} \to Y$ be a covering map, and let $f : X \to Y$ be a map such that a lift $\tilde{f} : X \to \tilde{Y}$ exists. Then the lift $\tilde{f}$ is the unique map lifting $f$.
   (f) There is no 3-sheeted cover of $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$.
   (g) There is no 4-sheeted cover of $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$.
   (h) Every cover of a Mobius band is regular.
   (i) Every cover of $S^1 \vee S^1$ is regular.
   (j) Every covering map $p : \tilde{X} \to X$ is the quotient map to the orbit space of the action of the deck group $G(\tilde{X})$ on $\tilde{X}$.
   (k) Let $\tilde{X} \to X$ be a connected covering space, and let $\tau$ be a deck map $\tilde{X} \to \tilde{X}$. If $\tau$ fixes a point, then $\tau$ is the identity.

(l) Let $X$ be a path-connected, locally path-connected, semi-locally simply-connected based space. Let $H$ be a subgroup of $\pi_1(X)$, and let $Y$ be a space with $\pi_1(Y) \cong H$. Then there exists a covering map $Y \to X$.

(m) Let $H_n(X)$ be the $n$th simplicial homology group of a $\Delta$-complex $X$. Then the rank of $H_n(X)$ is at most the number of $n$-simplices in $X$.

(n) Let $H_1(X)$ be the $n$th simplicial homology group of a 2-dimensional $\Delta$-complex $X$. Then $H_1(X)$ is free abelian.

(o) Let $H_2(X)$ be the $n$th simplicial homology group of a 2-dimensional $\Delta$-complex $X$. Then $H_2(X)$ is free abelian.