Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (n-sphere) (we sometimes view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

Practice problems

1. Let X be a topological space, and let $f, g : X \to S^n$ be two maps with the property that the points $f(x)$ and $g(x)$ are never antipodal for any $x \in X$. Prove that f and g are homotopic.
2. Suppose that a map $f : S^1 \to S^1$ is nullhomotopic. Show that f has a fixed point, and maps a point x to its antipode $-x$.
3. Let X and Y be topological spaces.
   (a) Suppose that Y is contractible. Prove that any two maps from X to Y are homotopic.
   (b) Suppose that X is contractible and Y is path-connected. Show that any two maps from X to Y are homotopic.
4. Show that, if X is contractible, then any retract of X is contractible.
5. (a) Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y .
   (b) Show that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y .
6. A topological space G with a group structure is called a *topological group* if the group multiplication map and inverse map

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \cdot h \end{aligned} \qquad\qquad \begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

   are continuous. Suppose G is a path-connected topological group. Show that, for each $g_0 \in G$, the left multiplication map

$$\begin{aligned} \ell_{g_0} : G &\longrightarrow G \\ g &\longmapsto g_0 \cdot g \end{aligned}$$

   is homotopic to the identity.
7. Let X be a CW complex. Show that any finite collection of cells in X are contained in a finite subcomplex.
8. Give an example (with proof) of a topological space that does not admit a CW complex structure.
9. Let X be a CW complex with n_d cells in dimension d, and let Y be a CW complex with m_d cells in dimension d. We described a natural CW complex structure on $X \times Y$. Count its cells in dimension d.
10. Let X be a CW complex, and let $q : \bigsqcup_{n,\alpha} D^n_\alpha \to X$ be the map defined by the characteristic maps. Show that q is a quotient map.
11. In this problem, we will show that CW complexes may not be metrizable. Let X be a 1-dimensional CW complex (that is, a graph).

(a) Let $x_1, x_2, \ldots,$ be a sequence of points in X, each in distinct edges. Explain why $(x_n)_n$ cannot converge.

(b) Suppose that v is a vertex in X with infinitely many edges incident on v. Explain why the weak topology on X cannot be metrizable.

12. **Definition (Isomorphism).** Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is called an *isomorphism* if there is a morphism $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$. In this case, we write $g = f^{-1}$ and call g the *inverse* of f.

(a) Suppose a morphism $f : X \to Y$ in a category $\mathscr{C}$ has an inverse g. Verify that the inverse is unique (so calling g "the" inverse is justified.)

(b) Show that an isomorphism is both monic and epic.

(c) Show that a map can be monic and epic, but not an isomorphism. *Hint:* Consider a category with only two objects.

(d) Let $f : X \to Y$ be a morphism in a category $\mathscr{C}$. Suppose there were morpihsms $g, h : Y \to X$ such that $f \circ g = id_Y$ and $g \circ h = id_X$. Show that $g = h$, and conclude that f is an isomorphism.

(e) Let $f : X \to Y$ be a map of topological spaces. Suppose there exists maps $g, h : Y \to X$ so that $f \circ g$ and $h \circ f$ are both homotopic to identity maps. Prove that f is a homotopy equivalence.

13. Show that a map homotopic to a homotopy equivalence is itself a homotopy equivalence.

14. Let $\underline{\text{Set}}$ be the category of sets, and for a set A let $\text{Hom}_{\underline{\text{Set}}}(-, A)$ denote the associated contravariant hom functor

$$\begin{aligned} \text{Hom}_{\underline{\text{Set}}}(-, A) : \mathscr{C} &\longrightarrow \underline{\text{Set}} \\ B &\longmapsto \text{Hom}_{\underline{\text{Set}}}(B, A) \\ [f : B \to C] &\longmapsto \left[f^* : \begin{array}{rl} \text{Hom}_{\underline{\text{Set}}}(C, A) & \to \text{Hom}_{\underline{\text{Set}}}(B, A)] \\ \phi & \mapsto \phi \circ f \end{array} \right] \end{aligned}$$

Prove that, if f is surjective, then f^* is injective.

15. Let $\underline{\text{Top}}_*$ be the category of based topological spacces and based maps. What is the coproduct of based spaces X and Y, along with the two associated maps? Prove your answer.

16. Let $\underline{\text{Top}}$ be the category of topological spaces, and let $P : \underline{\text{Top}} \to \underline{\text{Set}}$ be the map that takes a topological space X to its set of path components. Explain how to define P on morphisms to make it a functor, and verify that it is well-defined and functorial.

17. Recall that we defined a forgetful map

$$\begin{aligned} \underline{\text{Top}}_* &\longrightarrow \underline{\text{Top}} \\ (X, x_0) &\longmapsto X \\ f &\longmapsto f \end{aligned}$$

What is the "free functor" associated to this "forgetful functor"? For a topological space X, determine what universal property the "free based space on X" should satisfy, and describe what this topological space $F(X)$ (along with its basepoint, and distinguished map $X \to F(X)$) should be.

18. **Definition (Initial objects, terminal objects, zero objects).** An *initial object* I in a category $\mathscr{C}$, if it exists, is an object with the property that for any $X \in \mathscr{C}$, there is a unique morphism in $\mathscr{C}$ from I to X. Dually, an object T is a *terminal object* if for every $X \in \mathscr{C}$ there is a unique morphism $X \to T$. If an object is both initial and terminal, it is called a *zero object.*

(a) Show that, if an initial (or terminal, or zero) object exists in a category $\mathscr{C}$, it is unique up to unique isomorphism.

(b) Determine whether initial, terminal, or zero objects exists, and what they are, in the categories $\underline{\text{Set}}$, $\underline{\text{Ab}}$, $\underline{\text{Grp}}$, $\underline{\text{Top}}$, and $\underline{\text{Top}}_*$.

(c) Let $\mathscr{C}$ be a category and I an initial object in $\mathscr{C}$. Prove or disprove: If $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor, then $F(I)$ is an initial object in $\mathscr{D}$.

19. Let $F(S)$ denote the free group on the set S. Suppose that S_1 and S_2 are finite sets. Show that, if S_1 and S_2 have different cardinalities, then $F(S_1)$ and $F(S_2)$ are not isomorphic.
Hint: Show that $\mathrm{Hom}_{\underline{\text{Grp}}}(F(S), \mathbb{Q})$ has the structure of a $\mathbb{Q}$-vector space, and compute its dimension.

20. (a) Let G be a group and $[G, G]$ is commutator subgroup. Let $\phi : G \to G$ be an automorphism. Prove that $\phi([G, G]) \subseteq [G, G]$. (This means that $[G, G]$ is a *characteristic* subgroup.)

(b) Show that an automorphism $\phi : G \to G$ induces an automorphism $\tilde{\phi} : G^{ab} \to G^{ab}$.

(c) Let $\mathrm{Aut}(G)$ denote the group of automorphisms of G. Prove that the induced map

$$\Phi : \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(G^{ab})$$
$$\phi \longmapsto \tilde{\phi}$$

is a group homomorphism.

21. (a) Let X be a space, and let x_0 and x_1 be two points in the same path component of X. Given a path h from x_0 to x_1, let β_h be the associated change-of-basepoint map

$$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0).$$

Choose a (possibly different) path g from x_1 to x_0. Show that the automorphism

$$\beta_h \circ \beta_g : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is given by conjugation by an element of $\pi_1(X, x_0)$ (which one?). Such automorphisms are called *inner automorphisms*.

(b) Suppose that $\pi_1(X, x_0)$ is abelian. Show that the isomorphism

$$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$$

is independent of choice of path h.

22. We showed that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ is generated by the class of a loop α around a meridian circle and the class of a loop a longitudinal circle, as in Figure 1. Construct an explicit homotopy from $\alpha \cdot \beta$ to $\beta \cdot \alpha$.

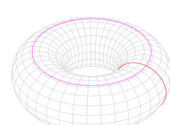

Figure 1: The loops α and β in Σ_1

23. (a) Suppose that $p_X : \tilde{X} \to X$ is a cover of a space X, and $p_Y : \tilde{Y} \to Y$ is a cover of a space Y. Show that $p_X \times p_Y : \tilde{X} \times \tilde{Y} \to X \times Y$ is a covering map.

(b) Conclude that $\mathbb{R}^n$ is a cover of the n-torus.

24. Suppose that $p_X : \tilde{X} \to X$ is a cover of a space X.

(a) Show that, if X is locally Euclidean then so is $\tilde{X}$.

(b) Show that, if X is Hausdorff, then so is $\tilde{X}$.

With this problem, we are most of the way to showing that, if X is a topological manifold, then so is $\tilde{X}$. We will need an extra condition on $\tilde{X}$, however, to ensure it is second-countable.

25. Let $f : X \to S^1$ be a continuous map. Show that, if f is nullhomotopic, then it factors through the covering map $p : \mathbb{R} \to S^1$.

26. Prove that A space X is simply-connected if and only if there is a unique homotopy class of paths connecting any two points in X .

27. Let $X \subseteq D^2$ be a subspace, and let $f : X \to X$ be a map without fixed points. Show that X is not a retract of D^2.

28. Show that there is no retraction from a Möbius band to its boundary.

29. Let $S^1 \times I$ be the cylinder, and suppose that $f : S^1 \times I \to X$ is a map that is constant on $S^1 \times \{1\}$. Show that the map f_* induced by f on fundamental group is trivial.

30. (a) Let $C_n \subseteq \mathbb{R}^2$ be the circle of radius n and center $(n, 0)$. Let $X = \cup_n C_n$. Show that $\pi_1(X)$ is the free group on countably many generators, with n^{th} generator a loop around C_n.

    (b) Show that X is not homeomorphic to the infinite wedge $\bigvee_{n\in\mathbb{N}} S^1$.
    Hint: Show that the point $(0, 0)$ in X has a countable neighbouhood basis, but the wedge point of $\bigvee_{n\in\mathbb{N}} S^1$ does not.

31. Let G and H be groups. Prove that $(G * H)^{ab} = G^{ab} \oplus H^{ab}$.

32. **True or counterexample.** For each of the following statements: if the statement is true, write "True". If not, state a counterexample. No justification necessary.
    Note: If the statement is false, you can receive partial credit for writing "False" without a counterexample.

    (a) Any contractible space is path-connected.

    (b) Any subspace of a contractible space is contractible.

    (c) Any quotient of a contractible space is contractible.

    (d) The product of two contractible spaces is contractible.

    (e) For any topological space X, any two maps $X \to S^\infty$ are homotopic.

    (f) For any topological space X, any two maps $\mathbb{R}^n \to X$ are homotopic.

    (g) Let X be a space and A a subspace. If A is a retract of X, then A and X are homotopy equivalent. (Note the distinction between *retract* and *deformation retract*).

    (h) A CW complex X is compact if and only if it is finite.

    (i) Any compact subset of a CW complex is closed.

    (j) Let F_S be the free group on a set S. Then every group isomorphism $F_S \to F_S$ is induced by a permutation of S.

    (k) There exists no non-abelian group with abelianization $\mathbb{Z}$.

    (l) If G is a finitely presented group, then its abelianization G^{ab} is finitely presented.

    (m) Every topological space is homeomorphic to the topological disjoint union of its connected components.

    (n) There does not exist a retraction from $\mathbb{R}^2$ to $\mathbb{R}^2 \setminus \{0\}$.

    (o) There does not exist a retraction from the torus $S^1 \times S^1$ to its meridian $S^1 \times \{(1, 0)\}$.

    (p) Let $X \subseteq Y$ be a subspace, and $x_0 \in X$. Then $\pi_1(X, x_0)$ is a subgroup of $\pi_1(Y, x_0)$.

(q) Suppose that X is a union of open, contractible subsets. Then $\pi_1(X) = 0$.

(r) Any presentation of a finite group must be finite.

(s) If G is a finitely presented group, then there exists a compact Hausdorff space with fundamental group isomorphic to G.

(t) If X is a connected graph, then $\pi_1(X)$ is a free group.

33. Compute the fundamental groups of the following spaces, and give presentations. Draw or describe the generators as loops in the space. Recall some tools we have developed:

- fundamental group is a homotopy invariant
- fundamental groups of Cartesian products and wedge sums
- the quotient of a CW complex by a contractible subcomplex is a homotopy equivalence
- Van Kampen theorem
- our results from Homework #4 on the effect on π_1 of gluing in disks along their boundaries
- our results from Homework #4 on the fundamental groups of CW complexes

(a) The product of $\mathbb{RP}^2$ and $\mathbb{CP}^2$

(b) The wedge sum of a cylinder and a Möbius band

Figure 2: The wedge sum of a cylinder and a Möbius band

(c) A genus-2 surface with two disks glued in, as in Figure 3

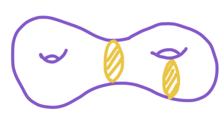

Figure 3: Two disks glued into a closed genus-2 surface

(d) A Klein bottle

(e) Two Möbius bands glued by the identity map along their boundaries.

(f) Each of graphs in Figure 4.

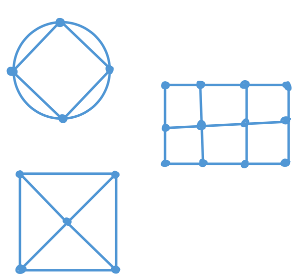

Figure 4: Three connected graphs

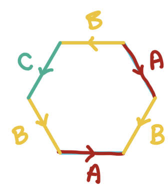

Figure 5: A 2-dimensional CW complex

(g) The CW complex shown in Figure 5.

(h) The plane $\mathbb{R}^2$ with n punctures

(i) The 2-sphere S^2 with n punctures

(j) $\mathbb{R}^3$ after deleting n lines through the origin

(k) The quotient of the plane $\mathbb{R}^2$ with n points identified to a single point.

(l) The torus with n punctures

(m) $\mathbb{R}\mathrm{P}^2$ with a puncture

(n) The quotient of the torus obtained by choosing an embedded disk D^2, and identifying its boundary S^1 to a single point

34. Describe the consruction of a CW complex with fundamental group is

$$\mathrm{SL}_2(\mathbb{Z}) \cong \langle a, b \mid abab^{-1}a^{-1}b^{-1}, (aba)^4 \rangle.$$

35. Let X be a finite CW complex. Explain why $\pi_1(X)$ must be a finitely presented group.

36. **(QR Exam, January 2016).** Let K be the complete graph on 4 letters, ie, K has 4 vertices, and there is a unique edge connecting each pair of distinct vertices.

    (a) Calculate $\pi_1(K)$.

    (b) Show that Σ_2 is not a deformation retract of any space homotopy equivalent to K.

37. **(QR Exam, September 2016).** Let

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \quad D = \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

Consider in the torus $T = S^1 \times S^1$ the homeomorph of the open disk

$$U = \{(x, y) \mid Re(x),\ Re(y) > 1/2\}.$$

Consider a homeomorphism $h : S^1 \to \partial U$ where ∂U is the boundary of the closure of U in T, and let $f : S^1 \to \partial U$ be the map given by $f(z) = h(z^2)$. Now let X be the quotient of

$$D \sqcup (T \setminus U)$$

by the smallest equivalence relation which has $z \sim f(z)$ for $z \in S^1 \subseteq D$. Find $\pi_1(X)$ in terms of generators and relations.

38. **(QR Exam, January 2017).** Prove that the usual inclusions $\mathbb{C}\mathrm{P}^0 \subseteq \mathbb{C}\mathrm{P}^1 \subseteq \cdots \subseteq \mathbb{C}\mathrm{P}^n$ define a CW filtration on $\mathbb{C}\mathrm{P}^n$.

39. **(QR Exam, September 2018).** Let

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Let X be the quotient of $S^1 \times [0,1]$ by the smallest equivalence relation $\sim$ satisfying

$$(x,0) \sim (e^{2\pi i/3}, 0),$$

$$(y,1) \sim (e^{2\pi i/6}, 1)$$

for $x, y \in S^1$. Calculate $\pi_1(X)$.

40. **(QR Exam, January 2018).** Let $Z = (\mathbb{C} \setminus \{e^{2k\pi i/5} \mid k \in \mathbb{Z}\}) \times [0,1]$. Let a space Y be obtained from Z by identifying $(z,0)$ with $(ze^{2\pi i/5}, 1)$ for every $z \in \mathbb{C} \setminus \{e^{2k\pi i/5} \mid k \in \mathbb{Z}\}$. Compute $\pi_1(Y)$.

41. **(QR Exam, January 2019).** Let S^1 be the unit circle in $\mathbb{C}$. Let Y by the space obtained from $S^1 \times [0,1] \times \{0,1\}$ (with the product topology, where each factor has the standard topology) by identifying $(z,0,0) \sim (z^3,0,1)$ and $(z,1,1) \sim (z^2,1,0)$. Calculate $\pi_1(Y)$.

42. **(QR Exam, August 2019).** Let X be a space obtained from three copies of the Möbius strip by attaching their boundaries homeomorphically. Calculate $\pi_1(X)$ in terms of generators and defining relations.