Name: ______________________ Score (Out of 4 points):

1. (4 points) Let Y be a (nonempty) path-connected topological space. Let n be a natural number. Show that any two continuous maps from $\mathbb{R}^n$ to Y are homotopic.

Solution. We prove on Homework #1 that homotopy is an equivalence relation. Thus, to prove the result, it suffices to prove the following two claims: Every map $\mathbb{R}^n \to Y$ is homotopic to a constant map, and any two constant maps $\mathbb{R}^n \to Y$ are homotopic.

Claim 1. *Every map $\mathbb{R}^n \to Y$ is homotopic to a constant map.*

Proof. Let $f : \mathbb{R}^n \to Y$ be any map. We will construct a homotopy from f to the constant map at $f(0)$. Define

$$F_t(x) = f((1-t)x).$$

Then F_t is continuous, as it is a composite of ccontinuous functtons. $F_0(x) = f(x)$ and $F_1(x) = f(0)$ for all x, so $F_t(x)$ is the desired homotopy. □

Claim 2. *Any two constant maps $\mathbb{R}^n \to Y$ are homotopic.*

Proof. Consider any two points $y_0, y_1 \in Y$. Because Y is path-connected, there exists some path $\gamma : I \to Y$ from y_0 to y_1. So define a homotopy

$$\begin{aligned} G_t : \mathbb{R}^n \times I &\longrightarrow Y \\ (x,t) &\longmapsto \gamma(t) \end{aligned}$$

Thus G_t is continuous, $G_0 : \mathbb{R}^n \to Y$ is the constant map at y_0, and $G_1 : \mathbb{R}^n \to Y$ is the constant map at y_1. Hence any two constant maps are homotopic. □

Remark: In fact, we can carry out this same argument if we replace $\mathbb{R}^n$ by any contractible space X. By definition of contractible, X must admit a homotopy H_t from id_X to a constant map at some point $x_0 \in X$. But then the composite of functions

$$X \times I \xrightarrow{H_t} X \xrightarrow{f} Y$$

is a homotopy from f to the constant map to $f(x_0)$. Because Y is path-connected, all constant maps $X \to Y$ are homotopic by the construction above.