Name: ______________________________ Score (Out of 4 points):

1. (4 points) Let Set be the category of sets and all functions. Let $\mathscr{C}$ be a reasonably nice[1] category. Fix an object A in $\mathscr{C}$. Define a map

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A,-) : \mathscr{C} &\longrightarrow \underline{\mathrm{Set}} \\ B &\longmapsto \mathrm{Hom}_{\mathscr{C}}(A,B) \end{aligned}$$

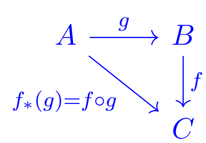
We can extend this map to a map of morphisms

$$\begin{aligned} \mathscr{C} &\longrightarrow \underline{\mathrm{Set}} \\ [f : B \to C] &\longmapsto [f_* : \mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{C}}(A,C)] \end{aligned}$$

to make it a covariant functor.[2] Explain how to define the map f_*, and verify that your construction is functorial.

Solution. We define

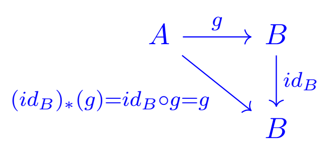
$$\begin{aligned} f_* : \mathrm{Hom}_{\mathscr{C}}(A,B) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(A,C) \\ g &\longmapsto f \circ g \end{aligned}$$



To check that this map is functorial, we need to check two properties: that it respects identity and that it respects composition.

Suppose $id_B : B \to B$ is the identity morphism. Then

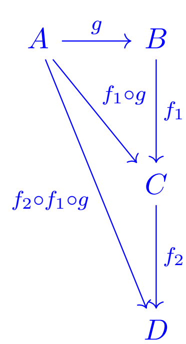
$$\begin{aligned} (id_B)_* : \mathrm{Hom}_{\mathscr{C}}(A,B) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(A,B) \\ g &\longmapsto id_B \circ g = g \end{aligned}$$



so $(id_B)_*$ is the identity map on $\mathrm{Hom}_{\mathscr{C}}(A,B)$ as desired.

Let $f_1 : B \to C$ and $f_2 : C \to D$. Then

$$\begin{aligned} (f_2 \circ f_1)_*(g) &= (f_2 \circ f_1) \circ g \\ &= f_2 \circ (f_1 \circ g) \\ &= (f_2)_*\big(f_1 \circ g\big) \\ &= (f_2)_*\big((f_1)_*(g)\big) \\ &= \big((f_2)_* \circ (f_1)_*\big)(g) \end{aligned}$$



so our functor respects composition.

[1] For set-theoretic reasons, we need $\mathscr{C}$ to be a *locally small category*. This holds for every category we will encounter.
[2] For each A there is a **covariant** functor $\mathrm{Hom}_{\mathscr{C}}(A,-)$: and a **contravariant** functor $\mathrm{Hom}_{\mathscr{C}}(-,A)$.