Name: ______________________________ Score (Out of 4 points):

1. (4 points) The *solid torus* is the space $D^2 \times S^1$, as in Figure 1. Prove that no retraction from a solid torus to its boundary can exist.

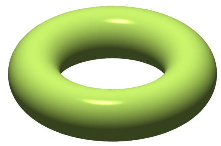

Figure 1: A solid torus

Your solution may use the strategy of Homework 3 Problem 1, but please give a complete argument, do not simply quote the results.

Solution. Recall that a *retraction* from a space X to its subspace A is a map $r : X \to A$ so that $r(a) = a$ for all $a \in A$. Let $\iota : A \to X$ be the inclusion of A. The definition of a retraction then states that $r \circ \iota = id_A$.

Let a_0 be a basepoint of A. Since the fundamental group is functorial, the induced maps must satisfy

$$r_* \circ \iota_* = (r \circ \iota)_* = (id_A)_* = id_{\pi_1(A,a_0)}.$$

Consequently, we can show that ι_* is injective:

$$\begin{aligned} & \iota_*(\alpha) = \iota_*(\beta) \\ \implies & r_*(\iota_*(\alpha)) = r_*(\iota_*(\beta)) \\ \implies & id_{\pi_1(A,a_0)}(\alpha) = id_{\pi_1(A,a_0)}(\beta) \\ \implies & \alpha = \beta. \end{aligned}$$

Now, consider the fundamental groups of the solid torus and its boundary, which is a torus. We proved that $\pi_1(D^2) = 0$ since D^2 is contractible, and that $\pi_1(S^1) = \mathbb{Z}$. We proved moreover that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ for any path-connected spaces X and Y. It follows that

$$\begin{aligned} \pi_1(D^2 \times S^1) &\cong \pi_1(D^2) \times \pi_1(S^1) \cong (0 \times \mathbb{Z}) \cong \mathbb{Z} \\ \pi_1(S^1 \times S^1) &\cong \pi_1(S^1) \times \pi_1(S^1) \cong (\mathbb{Z} \times \mathbb{Z}) \end{aligned}$$

However, there is no injective map from $\mathbb{Z}^2$ to $\mathbb{Z}$. All subgroups of $\mathbb{Z}$ are isomorphic to $\mathbb{Z}$ or $\{0\}$, and we know that $\mathbb{Z}$ and $\mathbb{Z}^2$ are not isomorphic, since $\mathbb{Z}^2$ has no cyclic generator.

We conclude, therefore, that no retraction from the solid torus to its boundary torus can exist.