Name: ______________________________ Score (Out of 3 points):

1. Let X and Y be path-connected, locally path-connected, semi-locally simply-connected spaces. Let $p_X : \tilde{X} \to X$ and $p_Y : \tilde{Y} \to Y$ be their universal covers.

   (a) (1 point) Explain why, for every continuous map $f : X \to Y$, there exists a continuous map $\tilde{f} : \tilde{X} \to \tilde{Y}$ that makes the following diagram commute.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

   Choose a basepoint x_0 for X and $y_0 = f(x_0)$ for Y. Let $\tilde{x_0} \in p_X^{-1}(x_0)$ and $\tilde{y_0} \in p_Y^{-1}(y_0)$. We proved that the universal cover $\tilde{X}$ is path-connected, and it is locally path-connected since it is locally homeomorphic to the locally path-connected space X. We can therefore apply our lifting theorem to the map

$$f \circ p_X : (\tilde{X}, \tilde{x_0}) \longrightarrow (Y, y_0).$$

   We proved $\pi_1(\tilde{X}, \tilde{x_0}) = 0$, so the image of π_1 under $(f \circ p_X)_*$ must be contained in $(p_Y)_*(\pi_1(\tilde{Y}, \tilde{y_0}))$. Our lifiting theorem therefore states that there is a lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ of the map $f \circ p_X$ mapping $\tilde{y_0}$ to $\tilde{x_0}$.

$$\begin{array}{ccccc} & & \overset{\tilde{f}}{\cdots\cdots\cdots\rightarrow} & & \tilde{Y} \\ & & & & \downarrow{\scriptstyle p_Y} \\ \tilde{X} & \xrightarrow{p_X} & X & \xrightarrow{f} & Y \end{array}$$

   The condition that $\tilde{f}$ is a lift of $(f \circ p_X)$ is exactly the statement that $p_Y \circ \tilde{f} = f \circ p_X$, in other words, the square above commutes.

   (b) (1 point) Is the map $\tilde{f}$ unique? Explain.

   The map $\tilde{f}$ is not in general unique. We chose a basepoint x_0 for X, and fixed a basepoint $\tilde{x_0}$ for $\tilde{X}$. Then for every choice of preimage point $\tilde{y_0} \in p_Y^{-1}(f(x_0))$, there is a lift $\tilde{f}$ mapping $\tilde{x_0}$ to $\tilde{y_0}$. By our uniqueness theorem, this single value of $\tilde{f}$ will uniquely determine the function $\tilde{f}$.

(c) (1 point) Consider the case that $X = S^1$ and $Y = S^1 \vee S^1$ as shown in Figure 1.

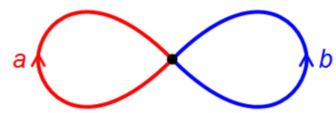

Figure 1: $Y = S^1 \vee S^1$

The universal cover of S^1 is $\mathbb{R}$, and the universal cover of $S^1 \vee S^1$ is shown in Figure 2.

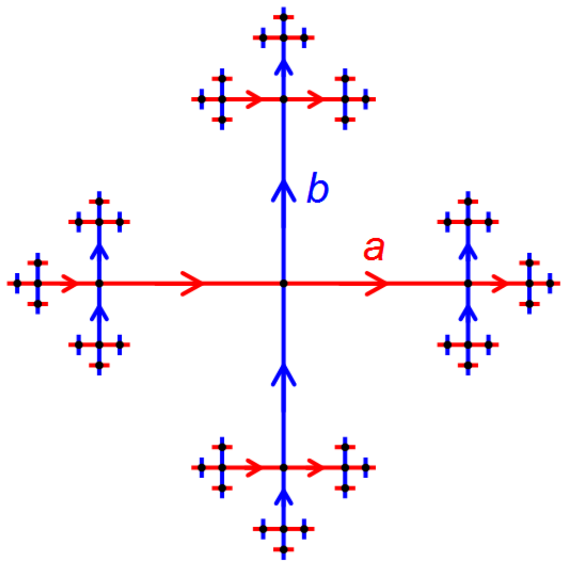

Figure 2: The universal cover $\tilde{Y}$ of $S^1 \vee S^1$

Let f be the constant-speed map that winds S^1 once (in the forward sense) around the loop a and then once (in the forward sense) around the loop b. Describe (informally) the corresponding map $\tilde{f}$ (or the set of all possible maps $\tilde{f}$) from $\mathbb{R}$ to the universal cover of $S^1 \vee S^1$.

Observe that, in the composition $\mathbb{R} \to S^1 \to S^1 \vee S^1$, the interval $[n, n+\frac{1}{2}]$ winds around the (red) edge a and the interval $[n+\frac{1}{2}, n+1]$ winds around the (blue) edge b for each $n \in \mathbb{N}$. We can compute the lift $\tilde{f}$ using our usual technique for lifting paths.

There is a map $\tilde{f}$ for every choice of vertex in the universal cover $\tilde{Y}$. Choose a vertex v, which we will make the image of 0. The map $\tilde{f}$ must then map the interval $[0, \frac{1}{2}]$ homeomorphically to the (red) edge labelled a directed away from v, then the interval $[\frac{1}{2}, 1]$ homeomorphically to the (blue) edge labelled b directed away from the next vertex. Then map $[1, \frac{3}{2}]$ to the outgoing red edge, $[\frac{3}{2}, 2]$ to the outgoing blue edge, etc, alternating between an outgoing red edge on $[n, n+\frac{1}{2}]$ and an outgoing blue edge on $[n+\frac{1}{2}, n+1]$ for each $n \in \mathbb{N}$.

Similarly, $\tilde{f}$ must extend to the negative real line by travelling backwards along the incoming blue edge on $[-n-\frac{1}{2}, -n]$, then backwards along the incoming red edge on $[-n-1, -n-\frac{1}{2}]$, for each $n \in \mathbb{N}$.