Name: ______________________________ Score (Out of 6 points):

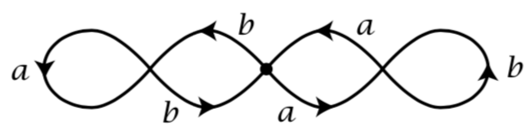
1. (6 points) Each of the following statements is either true or false. If it is true, write `True` and give a brief justification. If it is false, write `False` and give a brief justification or state a counterexample. You do not need to prove your counterexample. You may receive partial credit for correctly writing `True` or `False` with no justification.

   (a) Every connected cover of the torus is regular.

   True.

   We proved that the (based) connected covers of the torus T are in bijection with subgroups of $\pi_1(T, x_0)$, and that the cover corresponding to a subgroup H is regular if and only if H is normal in $\pi_1(T, x_0)$. Since $\pi_1(T, x_0)$ is abelian, all of its subgoups are normal.

   (b) Let $\tilde{X}$ be the cover of $S^1 \vee S^1$ shown below. We adopt our usual convention of identifying the fundamental group of $S^1 \vee S^1$ based at the wedge point x_0 with the free group $F_{\{a,b\}}$ on a, b. We choose a basepoint $\tilde{x_0}$ for $\tilde{X}$ (shown by a black dot) so that $H = p_*(\pi_1(\tilde{X}, x_0))$ is the subgroup of the free group on a, b freely generated by $a^2, b^2, aba^{-1}, bab^{-1}$.

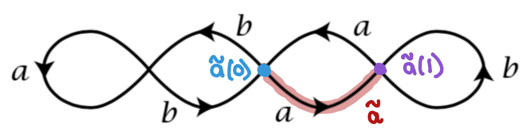


$\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$

   Then the element $a \in F_{\{a,b\}}$ is in the normalizer of H.

   False.

   Let $\tilde{a}$ denote the lift of a loop representing a to $\tilde{X}$, starting at the basepoint $\tilde{x_0}$. Then $\tilde{a}$ is illustrated below. By inspection, there is no graph automorphism taking $\tilde{x_0} = \tilde{a}(0)$ to $\tilde{a}(1)$. (For example, there is a self-loop of the graph incident on $\tilde{a}(1)$ but not $\tilde{a}(0)$.) Therefore a is not in the normalizer.



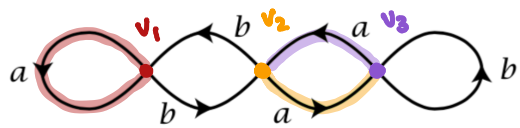
$\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$

   In fact, there is no graph automorphism taking $\tilde{x_0}$ to any of the other preimages of the wedge point (the three vertices), and so we see that $N(H) = H$, and the only elements in $N(H)$ are elements of $F_{\{a,b\}}$ that lift to loops in $\tilde{X}$ based at $\tilde{x_0}$.

(c) Again consider the cover $\tilde{X}$ of $S^1 \vee S^1$ shown above. Then the action of a on the preimage of x_0 is transitive.

False.

The preimage of x_0 are the three vertices v_1, v_2, v_3 shown below. By our definition of the action, our element a acts on v_i by mapping it to the endpoint of the lift of a loop representing a^{-1} starting at v_i. We see that a swaps v_2 and v_3, but fixes v_1. Hence the action is not transitive.



$\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$