Name: ______________________ Score (Out of 8 points):

1. (4 points) (a) (1 point) State the definition of the singular homology groups of a space X.

By definition, the singular homology groups $H_n(X)$ are the homology groups of the chain complex $(C_*(X), \partial)$ defined by

$$C_n(X) = \mathbb{Z}\langle\ \sigma \mid \sigma : \Delta^n \to X \text{ a continuous map}\rangle$$

and

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, v_1, \ldots, v_{i-1}, \hat{v_i}, v_{i+1}, \ldots, v_n]}$$

where $[v_0, v_1, \ldots, v_{i-1}, \hat{v_i}, v_{i+1}, \ldots, v_n]$ is the face of Δ^n that excludes the vertex v_i.

Concretely, $H_n(X) = \dfrac{\ker(\partial_n)}{\text{im}(\partial_{n+1})}$.

(b) (3 points) Let X be a single point. Working directly from the definition of singular homology, compute the singular homology groups of X.

Solution:

Fix n. Since X is a single point, there is single map $\sigma_n : \Delta^n \to X$, and its restriction to any face is the single map $\sigma_{n-1} : \Delta^{n-1} \to X$. Hence,

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} \sigma_{n-1}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

The associated chain complex is

$$\cdots \xrightarrow{\cong} \mathbb{Z}\sigma_{2n+1} \xrightarrow{0} \mathbb{Z}\sigma_{2n} \xrightarrow{\cong} \mathbb{Z}\sigma_{2n} \xrightarrow{0} \cdots \xrightarrow{\cong} \mathbb{Z}\sigma_1 \xrightarrow{0} \mathbb{Z}\sigma_0 \longrightarrow 0$$

We therefore have three cases:

$$H_{2n+1}(X) = \frac{\ker(\partial_{2n+1})}{\text{im}(\partial_{2n+2})} = \frac{\mathbb{Z}\sigma_{2n+1}}{\mathbb{Z}\sigma_{2n+1}} \cong 0 \qquad \text{for } n \geq 0.$$

$$H_{2n}(X) = \frac{\ker(\partial_{2n})}{\text{im}(\partial_{2n+1})} = \frac{0}{0} \cong 0 \qquad \text{for } n > 0.$$

$$H_0(X) = \frac{\ker(0)}{\text{im}(\partial_1)} = \frac{\mathbb{Z}\sigma_0}{0} \cong \mathbb{Z}.$$