Name: ______________________ Score (Out of 4 points):

1. (4 points) Let $S^1 \subseteq \mathbb{R}^4$ be the unit circle in $\mathbb{R}^2 \times \{0\}$. Compute the homology of the quotient $\mathbb{R}^4/S^1$.

Solution.

One approach tto this problem is to argue that the space $\mathbb{R}^4/S^1$ deformation retracts to $D^2/\partial D^2 \cong S^2$, and then use our result that homology is homotopy-invariant. In this solution, however, we will calculate the homology directly usign the long exact sequence of a pair.

Since S^1 is a closed smooth submanifold of $\mathbb{R}^4$, the pair $(\mathbb{R}^4, S^1)$ is a good pair. Thus

$$H_p(\mathbb{R}^4, S^1) = \widetilde{H}_p(\mathbb{R}^4/S^1)$$

The long exact sequence of a pair therefore has the form

$$\cdots \longrightarrow \widetilde{H}_p(S^1) \longrightarrow \widetilde{H}_p(\mathbb{R}^4) \longrightarrow \widetilde{H}_p(\mathbb{R}^4/S^1) \longrightarrow \widetilde{H}_{p-1}(S^1) \longrightarrow \cdots$$

By Homework 9 Problem 3(c), since $\mathbb{R}^4$ is contracible, its reduced homology groups $\widetilde{H}_p(\mathbb{R}^4)$ vanish for all p.

Moreover, we used the long exact sequence of the pair $(D^n, \partial D^n)$ to prove that

$$\widetilde{H}_p(S^1) = \begin{cases} \mathbb{Z}, & p = 1 \\ 0, & p \neq 1 \end{cases}.$$

Thus our long exact sequence has the form for $p > 2$,

$$\cdots \longrightarrow \widetilde{H}_p(\mathbb{R}^4) \longrightarrow \widetilde{H}_p(\mathbb{R}^4/S^1) \longrightarrow \widetilde{H}_{p-1}(S^1) \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow \widetilde{H}_p(\mathbb{R}^4/S^1) \longrightarrow 0 \longrightarrow \cdots$$

and for $p \leq 2$,

$$\cdots \longrightarrow \widetilde{H}_2(\mathbb{R}^4) \longrightarrow \widetilde{H}_2(\mathbb{R}^4/S^1) \longrightarrow \widetilde{H}_1(S^1) \longrightarrow \widetilde{H}_1(\mathbb{R}^4) \longrightarrow \widetilde{H}_1(\mathbb{R}^4/S^1) \longrightarrow \widetilde{H}_0(S^1) \longrightarrow \widetilde{H}_0(\mathbb{R}^4) \longrightarrow \widetilde{H}_0(\mathbb{R}^4/S^1) \longrightarrow 0$$

$$\cdots \longrightarrow 0 \longrightarrow \widetilde{H}_2(\mathbb{R}^4/S^1) \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \widetilde{H}_1(\mathbb{R}^4/S^1) \longrightarrow 0 \longrightarrow 0 \longrightarrow \widetilde{H}_0(\mathbb{R}^4/S^1) \longrightarrow 0$$

We conclude

$$\widetilde{H}_p(\mathbb{R}^4/S^1) = \begin{cases} \mathbb{Z}, & p = 2 \\ 0, & p \neq 2 \end{cases}.$$