Name: ______________________ Score (Out of 4 points):

1. (4 points) The *closed nonorientable surface N_g of genus g* is the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$,

$$N_g = \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2 \# \cdots \# \mathbb{R}\mathrm{P}^2.$$

It has a CW complex structure as follows [which you do not need to verify]:

- 1 zero-cell x
- g one-cells $a_1, a_2, \dots, a_g$
- 1 two-cell D attached along the word $a_1^2 a_2^2 a_3^2 \cdots a_g^2$.

Compute the cellular homology groups of N_g.

Solution.

The cellular chain complex associated to this cell structure on N_g is

$$\cdots \longrightarrow 0 \longrightarrow 0 \xrightarrow{\partial_3} \mathbb{Z}D \xrightarrow{\partial_2} \mathbb{Z}\langle a_1, \dots a_g \rangle \xrightarrow{\partial_1} \mathbb{Z}x \xrightarrow{\partial_0} 0.$$

Here,

- ∂_3, ∂_0 are the zero maps
- ∂_1 is the usual simplicial boundary map

$$\partial_1(a_i) = x - x \quad \text{for all } i,$$

  so ∂_1 is also the zero map.
- the differential ∂_2 is the map $\partial_2 : D \mapsto \sum_i n_i a_i$ where n_i is the degree of the map $S^1 \to S^1$ defined by

$$\partial D \xrightarrow{\text{attaching map}} (\text{1-skeleton of } N_g) \xrightarrow{\text{quotient map}} (\text{1-skeleton / complement of interior of } a_i).$$

  Since ∂D winds twice in the positive sense around each loop a_i, the degree is 2 for each i. So ∂_2 is the injective map

$$\partial_2 : D \mapsto 2a_1 + 2a_2 + \cdots + 2a_g.$$

Thus

$$H_0(X) = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\mathbb{Z}x}{0} \cong \mathbb{Z}$$

$$H_1(X) = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\mathbb{Z}\langle a_1, \dots, a_g \rangle}{\mathbb{Z}\langle 2a_1 + 2a_2 + \cdots 2a_g \rangle}$$

$$= \frac{\mathbb{Z}\langle a_1, \dots, a_{g-1}, a_1 + a_2 + \cdots + a_g \rangle}{\mathbb{Z}\langle 2a_1 + 2a_2 + \cdots + 2a_g \rangle} \qquad \text{[by change of basis]}$$

$$\cong \mathbb{Z}^{g-1} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$$

$$H_2(X) = \frac{\ker(\partial_2)}{\operatorname{im}(\partial_3)} = \frac{0}{0} = 0.$$