Name: ______________________ Score (Out of 5 points):

1. (5 points) Let (X, x_0) and (Y, y_0) be based spaces, and suppose $(X, \{x_0\})$ and $(Y, \{y_0\})$ are good pairs. Use the Mayer–Vietoris sequence to give a new calculation of the homology of the wedge product $X \vee Y$ (defined by identifying x_0 and y_0) in terms of the homology of X and Y. [You may describe the result directly, without using Mayer–Vietoris, in homological degree 0].

Solution.

Since $(X, \{x_0\})$ and $(Y, \{y_0\})$ are good pairs, we can find a neighbourhood $V_{x_0} \subseteq X$ that deformation retracts to x_0, and a neighbourhood $V_{y_0} \subseteq Y$ that deformation retracts onto y_0.

Let A be the image of $X \cup V_{y_0}$ in $X \vee Y$, and let B be the image of $V_{x_0} \cup Y$ in $X \vee Y$. Then A deformation retracts onto X and B deformation retracts onto Y. Thus $H_*(A) = H_*(X)$ and $H_*(B) = H_*(Y)$.

[Students are not expected to check this point explicitly, but we can construct this deformation retraction as follows. Let $F_t : V_{y_0} \to V_{y_0}$ be the deformation retraction to y_0. Then we can extend F_t to a deformation retraction $F'_t : (X \cup V_{y_0}) \to (X \cup V_{y_0})$ from $(X \cup V_{y_0})$ to $(X \cup \{y_0\})$ by defining F'_t to be the identity on X for all t. We can then use the universal property of the quotient topology to verify that F'_t induces a well-defined deformation retraction on the image A of $(X \cup V_{y_0})$ in $X \vee Y$.]

The subsets A and B are open in $X \vee Y$ and cover $X \vee Y$, so we may apply the Mayer–Vietoris sequence to the decomposition $X \vee Y = A \cup B$.

The intersection $A \cap B$ is the image of $V_{x_0} \cup V_{y_0}$ in $X \vee Y$, which deformation retracts onto the wedge point in $X \vee Y$. Thus $\widetilde{H}_n(A \cap B) = 0$ for all n.

The Mayer–Vietoris sequence has the form

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(A \cup B) \xrightarrow{\delta} H_{n-1}(A \cap B) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_0(A \cup B) \longrightarrow 0.$$

In this instance, when $n \geq 2$, $H_n(A \cap B) = H_{n-1}(A \cap B) = 0$, so we have an exact sequence

$$\cdots \longrightarrow 0 \xrightarrow{\Phi} H_n(X) \oplus H_n(Y) \xrightarrow{\Psi} H_n(X \vee Y) \xrightarrow{\delta} 0 \longrightarrow \cdots$$

and $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for $n \geq 2$. The tail of the sequence has the form

$$\cdots \longrightarrow 0 \xrightarrow{\Phi} H_1(X) \oplus H_1(Y) \xrightarrow{\Psi} H_1(X \vee Y)$$
$$\xrightarrow{\delta} H_0(A \cap B) \xrightarrow{\Phi} H_0(X) \oplus H_0(Y) \xrightarrow{\Psi} H_0(X \vee Y) \longrightarrow 0.$$

The generator 1 of $H_0(A \cap B) \cong \mathbb{Z}$ corresponds to the single path component of $A \cap B$. It is mapped by $\Phi : H_0(A \cap B) \to H_0(X) \oplus H_0(Y)$ to the element $([x_0], -[y_0])$ corresponding to the path components of $x_0 \in X$ and $y_0 \in Y$, respectively. This map is injective, hence by exactness δ is zero and $H_1(X \vee Y) \cong H_1(X) \oplus H_1(Y)$.

In degree zero, $H_0(X \vee Y)$ is free abelian of rank one smaller than $H_0(X) \oplus H_0(Y) \cong H_0(X \sqcup Y)$, since $X \vee Y$ has one fewer path components than $X \sqcup Y$: the path components of x_0 and y_0 are merged in the quotient $X \vee Y$.