Terms and concepts covered: Homotopy; homotopic maps; nullhomotopic map. Homotopy rel a subspace. Homotopy equivalence; homotopy type; contractible. Deformation retraction. CW complex; weak topology. Products, wedge sums, and quotients of CW complexes.

Corresponding reading: Hatcher, Chapter 0, "Homotopy and homotopy type", "Cell complexes", "Operations on spaces" & Hatcher, Appendix, through Prop A.3.

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. **(Point-set review).**
   (a) Show by example that the inverse of a continuous, bijective function need not be continuous.
   (b) Prove that the continuous image of a connected set is connected.
   (c) Prove that the continuous image of a path-connected set is path-connected.
   (d) Prove that the continuous image of a compact set is compact.
   (e) Show by example that the preimage of a compact (respectively, connected, path-connected) under a continuous function need not be compact (respectively, connected, path-connected).
   (f) Show that a closed subset of a compact set is compact.
   (g) Show that, in a Hausdorff space, compact subsets are closed.
   (h) Show by example that compact sets need not be closed in general. *Hint:* consider finite topological spaces.

2. Let X be a topological space, and let $f, g : X \to \mathbb{R}^n$ be continuous maps. Show that f and g are homotopic via the homotopy
$$F_t(x) = t\,g(x) + (t-1)f(x).$$

3. Let X be a topological space. Show that all constant maps to X are homotopic if and only if X is path-connected. In general, what are the homotopy classes of constant maps in X?

4. Recall that a subset $S \subseteq \mathbb{R}^n$ is *star-shaped* if there is a point $x_0 \in S$ such that, for any $x \in S$, the line segment from x_0 to x is contained in S. Show that any star-shaped subset of $\mathbb{R}^n$ is contractible. Conclude in particular that convex subsets of $\mathbb{R}^n$ are contractible.

5. Let X be a space and let $A \subseteq X$ be a deformation retract. Verify that X and A are homotopy equivalent.

6. Prove that every contractible space is path-connected. *Hint:* From a homotopy $F_t(x)$ we obtain, for each fixed x, a continuous function $t \mapsto F_t(x)$.

7. Find the mistake in the following "proof" that every path-connected space is contractible.

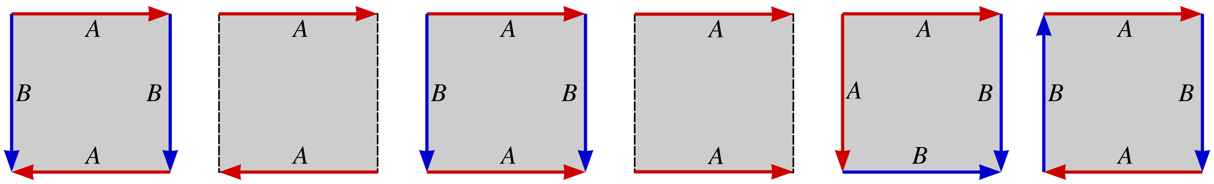
   False proof. Let X be a path-connected space, and choose a basepoint $x_0 \in X$. Then for each $x \in X$, there is some path $\gamma_x : [0,1] \to X$ from x to x_0. Define a homotopy
$$F_t(x) = \gamma_x(t).$$
   Then F_t is a homotopy from the identity id_X to the constant map at x_0. We conclude that X is contractible.

8. Let $S^1 \subseteq \mathbb{R}^2$ be the unit circle. Find the mistake in the following "proof" that S^1 is contractible.

   False proof. There is a deformation retraction from S^1 to the point $(1,0)$ given by the homotopy
$$F_t(x,y) = \frac{(1-t)(x,y) + t(1,0)}{||(1-t)(x,y) + t(1,0)||}.$$

9. Suppose X and Y are homotopy equivalent spaces. Show that Y is path-connected if and only if X is.

10. **(Quotient surfaces).** Identify among the following quotient spaces: a cylinder, a Möbius band, a sphere, a torus, real projective space, and a Klein bottle.



11. Let $X = \bigcup_n X^n$ be a CW complex with n-skeleton X^n. Recall that we defined the topology on X so that a set U is open iff $U \cap X^n$ is open for every n.
    (a) Suppose that X is finite-dimensional, that is, $X = X^N$ for some N. Show that the topology on X agrees with the topology from our inductive definition of the N-skeleton X^N as a quotient space.
    (b) Again let X be any CW complex. Show that a set $C \subseteq X$ is closed iff $C \cap X^n$ is closed for every n.

12. We define a *subcomplex* of a CW complex X to be a closed subset that is equal to a union of cells. Show that a subcomplex A is itself a CW complex, by verifying inductively that the images of the attaching map of an n-cell in A must be contained in its $(n-1)$-skeleton A^{n-1}.

13. Verify the details of the natural CW complex structure on a product of CW complexes, or a quotient of a CW complex by a subcomplex.

14. Let X be a CW complex, and X^n its n-skeleton. Verify that the weak topology on X^n agrees with the subspace topology on X^n as a subspace of X.

15. Let X be a CW complex. See Assignment Problem 5 for the definition of the *characteristic maps* Φ_α. Verify that, for any α, the map Φ_α is continuous.

16. Prove that any finite CW complex is compact, by realizing it as the continuous image of a finite union of closed balls.

17. (a) Let X be a CW complex. See Assignment Problem 5 for the definition of the *characteristic maps* Φ_α. For a subset $A \subseteq X$, prove that A is open (respectively, closed) if and only if $\Phi_\alpha^{-1}(A)$ is open (respectively, closed) for every α.
    (b) Conclude that
    $$\sqcup \Phi_\alpha : \bigsqcup_{n,\alpha} D^n_\alpha \to X$$
    is a quotient map.

Assignment questions

(Hand these questions in! Questions labelled "bonus" are optional.)

1. **(Homotopy defines an equivalence relation).**
    (a) Prove the following lemma.

    Lemma (Pasting Lemma). Let A, B be a topological spaces, and suppose A is the union $A = A_1 \cup A_2$ of closed subsets A_1 and A_2. Then a map $f : A \to B$ is continuous if and only if its restrictions $f|_{A_1}$ and $f|_{A_2}$ to A_1 and A_2, respectively, are continuous.

    (b) Let X, Y be topological spaces and consider the set of continuous maps $X \to Y$. Show that the relation "f is homotopic to g" defines an equivalence relation on this set.

2. **(Homotopy equivalence is an equivalence relation).** Show that "homotopy equivalence" defines an equivalence relation on topological spaces.

3. **(S^∞ is contractible).** Define the infinite-dimensional sphere S^∞ as the space

$$S^\infty = \bigcup_{n\geq 0} S^n = \left\{ (x_1, x_2, \dots,) \;\middle|\; x_i \in \mathbb{R},\ x_i = 0 \text{ for all but finitely many } i, \sum_i x_i^2 = 1 \right\}$$

It is topologized so that a subset U is open if and only if $U \cap S^n$ is open for every n.
Show that S^∞ is contractible. *Hints:*

- The map $S^\infty \to S^\infty$ given by $(x_1, x_2, x_3, \dots) \longmapsto (0, x_1, x_2, x_3, \dots)$ is continuous.
- Warm-up Problem 8.

4. **(A CW complex structure on the sphere).** Let S^n denote the n-sphere. In general we understand S^n is defined up to homeomorphism, but for the purposes of this question we will concretely define S^n to be the unit sphere in $\mathbb{R}^{n+1}$ with the Euclidean topology,

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid |\mathbf{x}| = 1\}.$$

Let D^n denote the closed n-ball. Again, to be concrete we take D^n to be the unit ball in $\mathbb{R}^n$,

$$D^n = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1\}.$$

(a) Prove the following theorem.

> **Theorem (A homeomorphism criterion).** Let $f : X \to Y$ be a continuous, bijective map of topological spaces. Suppose X is compact and Y is Hausdorff. Then f is a homeomorphism.

(b) Let X be a topological space, and let $\sim$ be an equivalence relation on X. Let $X/\sim$ denote the corresponding quotient space, and $q : X \to X/\sim$ the quotient map. State the definition of the quotient topology on $X/\sim$, and state the universal property of the quotient map q.

(c) Let $D^n/\sim$ be the quotient of D^n obtained by identifying all points in the boundary to a single point. Use parts (a) and (b) to prove that $D^n/\sim$ is homeomorphic to S^n.

Hint: Consider the map

$$f : D^n \longrightarrow S^n$$

$$x = (x_1, x_2, \dots, x_n) \mapsto \begin{cases} \left(\left(2\sqrt{\frac{1}{||x||} - 1}\right) x_1, \dots, \left(2\sqrt{\frac{1}{||x||} - 1}\right) x_n, 1 - 2||x||\right), & \text{if } x \neq 0 \\ (0, 0, \dots, 0, 1), & \text{if } x = 0 \end{cases}$$

Remark: Going forward, you may assert without proof the identity of quotient spaces such as this one and the ones in Warm-Up Problems 10 and **??**. But we should check this rigorously at least this once!

5. **(CW complexes are Hausdorff).**

> **Definition (The characteristic map).** Let X be a CW complex. For each n-cell e^n_α the associated *characteristic map* Φ_α is the composition
>
> $$\Phi_\alpha : D^n_\alpha \hookrightarrow X^{n-1} \bigsqcup_\beta D^n_\beta \longrightarrow X^n \longrightarrow X$$
>
> Specifically $\Phi_\alpha|_{\partial D^n}$ is the attaching map ϕ_α, and Φ_α maps the interior of D^n homeomorphically to e^n_α.

(a) Let X be a topological space. Recall that topologists say "points are closed" in X to mean that the singleton set $\{x\}$ is closed for all $x \in X$. Prove that, in a CW complex, points are closed.
Note: Hatcher writes (p522), "Points are closed in a CW complex X since they pull back to closed sets under all characteristic maps Φ_α." If you quote this statement, you must explain why points pull back to closed sets, and explain why this observation implies that points are closed.

(b) Prove that a CW complex is Hausdorff. *Hint:* Read the first half of Hatcher p522, and explain the proof of Proposition A.3 in the special case that A and B are points. You may use the book as a reference while you write this proof, though you should not simply copy the book!

6. **(Compact subsets of CW complexes and the closure-finite property).**

(a) Let X be a CW complex, and let S be a (possibly infinite) subset of X such that every point of S is in a distinct cell of X. Prove that S is closed. Since the same argument applies to any subset of S, conclude that S has the discrete topology.

(b) Prove the following lemma.

> **Lemma (Compact subsets of CW complexes).** Let X be a CW complex. Any compact subset of X intersects only finitely many cells.

(c) Show that the closure of e^n_α in X is equal to the image of the characteristic map Φ_α (Question 5).

(d) "CW" stands for "closure-finiteness, weak topology". Prove the "closure-finiteness" property.

> **Proposition (Closure-finiteness of CW complexes).** Let X be a CW complex. The closure of any cell intersects only finitely many other cells.

(e) The *infinite earring* is a subspace of $\mathbb{R}^2$ defined as the union $\bigcup_{n\geq 1} C_n$, where C_n is the circle of radius $\frac{1}{n}$ and center $(\frac{1}{n}, 0)$. See Figure 2. It is a favourite source of counterexamples in algebraic topology.[1]

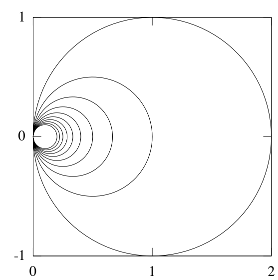


Figure 2: The infinite earring

Use part (b) to show that the topology on the infinite earring does not agree with the weak topology on a countable wedge of circles.
Remark: In fact, the infinite earring is not even homotopy equivalent to a CW complex.

7. **(Bonus: Homotopies as paths of maps).** Let X and Y be locally compact, Hausdorff topological spaces. Consider the space $C(X, Y)$ of continuous maps from X to Y with the compact-open topology. Let I be a closed interval. Show that the definition of a homotopy of maps $X \to Y$ is equivalent to the definition of a continuous map $I \to C(X, Y)$. In other words, a homotopy is a path through the space of continuous maps.

[1] The infinite earring is often called the *Hawaiian earring*, but there are concerns that this term is culturally insensitive, so I am trying to train myself to stop using it. Hatcher calls the space the *shrinking wedge of circles*. I do not love this term either, since, as you will prove, it is not homeomorphic to a wedge of circles.