Terms and concepts covered: Singular homology is a homotopy invariant. Singular homology is functorial. Homology of S^n and the Brouwer fixed point theorem for D^n. Diagram chases; the short five lemma. Relative chains, relative cycles, relative boundaries. Relative homology groups. Long exact sequence of a pair. "A short exact sequence of chain complexes induces a long exact sequence of homology groups". Excision theorem. $H_n(X, A) \cong \widetilde{H}_n(X/A)$ for a good pair (X, A). Five lemma, singular $\cong$ simplicial homology, degree of a map $S^n \to S^n$, properties of degree.

Corresponding reading: Hatcher Ch 2.1, Exact sequences and excision, "The Equivalence of Simplicial and Singular Homology", Ch 2.2, "Degree", Ch 2.A Homology and fundamental group.

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$
be a short exact sequence of abelian groups. The group B is called an *extension* of C by A. The question of classifying the possible groups B given the groups A and C is called the *extension problem*. Solve the extension problem for the following short exact sequences.
$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow ? \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow ? \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z} \longrightarrow ? \longrightarrow \mathbb{Z} \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z} \longrightarrow ? \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow ? \longrightarrow \mathbb{Z} \longrightarrow 0$$

2. We proved in class that $\widetilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$
   (a) Conclude that S^n and S^m are not homotopy equivalent unless $n = m$.
   (b) Prove that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.
   Hint: Consider a hypothetical homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^m$ and its restriction to $\mathbb{R}^n \setminus \{0\}$.

3. Let $A \subseteq X$ be spaces Recall that we defined $C_n(X, A)$ as the quotient of singular chain groups $C_n(X)/C_n(A)$.
   (a) Explain why we can identify $C_n(X, A)$ with free abelian group of singular n-chains $\Delta^n \to X$ with image not fully contained in A.
   (b) Explain why we did not define $C_n(X, A)$ as the subgroup of $C_n(X)$ spanned by singular n-chains $\Delta^n \to X$ with image not fully contained in A.
   Hint: Is this a sub-chain complex? What happens when you restrict ∂?

4. Let X be a space, and $* \in X$.
   (a) Show that
   $$H_n(X, X) = 0 \qquad \text{for all } n.$$
   (b) Show that
   $$H_n(X, \varnothing) \cong H_n(X) \qquad \text{for all } n.$$
   (c) Show that
   $$H_n(X, *) \cong \widetilde{H}_n(X) \qquad \text{for all } n.$$

5. Suppose that (X, A) is a good pair of spaces, and that A is contractible. Use the long exact sequence of a pair to prove that
$$\widetilde{H}_n(X) \cong \widetilde{H}_n(X/A) \qquad \text{for all } n.$$

6. Let $f : (X, A) \to (Y, B)$ be a map of pairs (Assignment Problem 3). Verify that the induced map on relative homology is functorial.

7. Let $f : X \to Y$ be a nullhomotopic map. Show that the induced map $f_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ is zero for all n. What are the induced maps $f_! H_n(X) \to H_n(Y)$?

Assignment questions

(Hand these questions in!)

1. (a) Verify that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in every degree.
   (b) Verify that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ are not homotopy equivalent. Conclude that, although homology groups are homotopy invariants, they are not *complete* invariants (i.e., they are not sufficient to distinguish all homotopy types).

2. In this problem, we will prove one of the foundational results of homological algebra. You may consult Hatcher p116-118 as much as you would like while you write your proof, but you should make a good-faith effort to attempt the diagram chases on your own!

   Slogan: A short exact sequence of chain complexes gives rise to a long exact sequence of homology groups.

   The complete theorem statement is as follows.

   Theorem (The LES on homology of an SES of chain complexes). Let (A_*, d^A), (B_*, d^B) and (C_*, d^C) be chain complexes. Let
$$0 \longrightarrow A_* \xrightarrow{i_\#} B_* \xrightarrow{j_\#} C_* \longrightarrow 0$$
   be a short exact sequence of these chain complexes. In other words, $i_\#$ and $j_\#$ are chain maps such that the sequence of group homomorphisms
$$0 \longrightarrow A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \longrightarrow 0$$
   is exact for all n. Then there exists a long exact sequence on homology groups
$$\cdots \longrightarrow H_n(A_*) \xrightarrow{i_*} H_n(B_*) \xrightarrow{j_*} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{i_*} H_{n-1}(B_*) \xrightarrow{j_*} H_{n-1}(C_*) \longrightarrow \cdots$$

   The map δ is called the *connecting homomorphism*.

   Remark: This result is called the *Zig-Zag Lemma* or *Snake Lemma*, due to the shape of the connecting homomorphism when the result is formulated as in this diagram.

   Remark: Implausibly, a construction of the connecting homomorphism appeared at the beginning of a 1980 romantic comedy. You can watch the clip here. The film was nominated for a Razzie Award for Worst Screenplay.

   (a) The following commutative diagram shows our short exact sequence of chain complexes.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{d^A_{n+1}} & A_n & \xrightarrow{d^A_n} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow i_\# & & \downarrow i_\# & & \downarrow i_\# & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow{d^B_{n+1}} & B_n & \xrightarrow{d^B_n} & B_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow j_\# & & \downarrow j_\# & & \downarrow j_\# & & \\
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{d^C_{n+1}} & C_n & \xrightarrow{d^C_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Given a cycle $c \in C_n$, explain how to construct a cycle $\delta(c)$ in A_{n-1}, and verify that your map is well-defined as a map on homology $\delta : H_n(C_*) \to H_{n-1}(A_*)$.

(b) Verify that δ is a group homomorphism.

(c) Verify that the sequence

$$\cdots \longrightarrow H_n(A_*) \xrightarrow{i_*} H_n(B_*) \xrightarrow{j_*} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{i_*} H_{n-1}(B_*) \xrightarrow{j_*} H_{n-1}(C_*) \longrightarrow \cdots$$

is exact.

(d) Given a space X, let $\widetilde{C}_*(X)$ denote the augmented singular chain complex of X. For space $A \subseteq X$, let $i : A \to X$ denote the inclusion map. Verify that the following are short exact sequences of chain complexes

$$0 \longrightarrow C_*(A) \xrightarrow{i_\#} C_*(X) \xrightarrow{j_\#} C_*(X, A) \longrightarrow 0$$

$$0 \longrightarrow \widetilde{C}_*(A) \xrightarrow{i_\#} \widetilde{C}_*(X) \xrightarrow{j_\#} C_*(X, A) \longrightarrow 0$$

where $j_\#$ is the quotient map $C_*(X) \to C_*(X)/C_*(A) = C_*(X, A)$. Deduce the two versions of the long exact sequence of a pair.

(e) Verify the following comment of Hatcher (p117):

> If a class $[\alpha] \in H_n(X, A)$ is represented by a relative cycle α, then $\delta[\alpha]$ is the class of the cycle $\partial\alpha$ in $H_{n-1}(A)$.

For this reason, in the long exact sequence of a pair the connecting homomorphism is sometimes called the *boundary map*.

3. (a) **Definition (Map of pairs).** Let $A \subseteq X$ and $B \subseteq Y$ be spaces. A *map of pairs* $(X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$.

Verify that a map of pairs $f : (X, A) \to (Y, B)$ induces a chain map $f_\sharp : C_*(X, A) \to C_*(Y, B)$. It follows that it induces a map on homology (see Warm-up Problem 6).

(b) Let $B \subseteq A \subseteq X$ be spaces. Show that there is a short exact sequence of chain complexes

$$0 \longrightarrow C_*(A, B) \longrightarrow C_*(X, B) \longrightarrow C_*(X, A) \longrightarrow 0$$

(c) Write down the corresponding long exact sequence on homology groups. It is called the *long exact sequence of a triple*.

(d) By modifying our proof of the analogous statement for absolute homology, we can prove the following. (You do not need to check this).

> **Proposition (Homotopic maps of pairs induce the same map on relative homology).** Suppose two maps of pairs $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs. Then for all n, the induced maps $f_*, g_* : H_n(X, A) \to H_n(Y, B)$ are equal.

Let $B \subseteq A \subseteq X$, and suppose that A deformation retracts to B. Show that $H_n(A, B) = 0$ for all n, and use the long exact sequence of a triple to conclude that $H_n(X, B) \cong H_n(X, A)$ for all n.
Remark: We use this result in the proof that $H_n(X, A) \cong \widetilde{H}_n(X/A)$ when (X, A) is a good pair.

Remark: We will not prove the following property, but it is useful to know:

Proposition (Naturality of a map of pairs). Let $f : (X, A) \to (Y, B)$ be a map of pairs. Then the long exact sequence of a pair is *natural* in the sense that the following diagram commutes.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X,A) & \xrightarrow{\delta} & H_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\ \cdots & \longrightarrow & H_n(B) & \xrightarrow{i_*} & H_n(Y) & \xrightarrow{j_*} & H_n(B,Y) & \xrightarrow{\delta} & H_{n-1}(B) & \longrightarrow & \cdots \end{array}$$

4. In this problem, we will complete our proof of the following theorem.

Theorem ($H_1(X) \cong \pi_1(X, x_0)^{ab}$). Let X be path-connected space with basepoint x_0. There is a surjective group homomorphism

$$\begin{aligned} h : \pi_1(X, x_0) &\longrightarrow H_1(X) \\ [\gamma] &\longmapsto \text{singular 1-chain } \gamma \end{aligned}$$

whose kernel is the commutator subgroup of $\pi_1(X, x_0)$. In particular,

$$H_1(X) \cong \pi_1(X, x_0)^{ab}.$$

Last week, we proved that the Hurewicz map h is a group homomorphism, that h surjects, and that the kernel of h contains the commutator subgroup of $\pi_1(X, x_0)$. It remains to show that the commutator subgroup contains $\ker(h)$.
You may read Hatcher 2.A and other relevant sections while you write your solutions.

(a) Read the three paragraphs in Hatcher before Proposition 2.6 , starting at the bottom of p108 with

> "Cycles in singular homology are defined algebraically, but they can be given a somewhat more geometric interpretation . . ."

Explain in your own words the construction K_ξ in the case $n = 2$, and why "elements of $H_2(X)$ are represented by maps of closed oriented surfaces into X."

(b) Let $[\gamma] \in \pi_1(X, x_0)$ be a loop such that $h([\gamma]) = 0$. Our goal is to show that $[\gamma]$ is in the commutator subgroup of $\pi_1(X, x_0)$. Since $h([\gamma]) = 0$, the loop γ (viewed as a 1-cycle) is the boundary of some 2-chain $\sum_i n_i \sigma_i$. Let $\tau_{i,j}$ denote the singular 1-simplices defined by

$$\partial \sigma_i = \tau_{i,0} - \tau_{i,1} + \tau_{i,2}$$

thus

$$\gamma = \partial \left(\sum_i n_i \sigma_i \right) = \sum_{i,j} (-1)^j n_i \tau_{i,j}.$$

Explain why we can group all the 1-simplices into cancelling pairs, with one remaining term $\tau_{i,j}$ equal to γ.

(c) Explain why we can associate the chain $\sum_i n_i \sigma_i$ to a map $\sigma : K \to X$ from a 2-dimensional Δ-complex K with one boundary loop corresponding to γ.

(d) Explain how we can use the homotopy extension property (Homework 3 Problem 4(d)) to homotope σ (rel the edge corresponding to γ) so that all of its vertices map to x_0.

(e) Deduce that we obtain a new map $\sigma' : K \to X$ associated to a new 2-chain $\sum_i n_i \sigma'_i$ with

$$\gamma = \partial \left(\sum_i n_i \sigma'_i \right) = \sum_{i,j} (-1)^j n_i \tau'_{i,j}$$

and every 1-simplex $\tau'_{i,j}$ a loop based at x_0.

(f) Let $[[\tau'_{i,j}]]$ denote the image of $\tau'_{i,j}$ in $\pi_1(X, x_0)^{ab}$. Explain why $\sum_{i,j}(-1)^j n_i [[\tau'_{i,j}]]$ is zero in $\pi_1(X, x_0)^{ab}$. *Hint:* The boundary of a 2-simplex is zero in π_1.

(g) Conclude that γ is zero in $\pi_1(X, x_0)^{ab}$, hence, $[\gamma]$ is in the commutator subgroup. This concludes our proof of the Hurewicz theorem for π_1.

(h) Let $f : (X, x_0) \to (Y, f(x_0))$ be a continuous map of topological spaces. Verify that the following diagram commutes,

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, f(x_0)) \\ \Big\downarrow h & & \Big\downarrow h \\ H_1(X) & \xrightarrow{f_*} & H_1(Y) \end{array}$$

and explain how this allows us to compute the map induced by f on H_1 using the map induced by f on π_1.

Remark: The commutativity of this diagram is the statement that the Hurewicz homomorphism is a *natural transformation* from the functor π_1 to the functor H_1 [redefined to be a functor Top$*$ $\to$ Grp].

5. **Definition (Local homology groups).** Let X be a space. The *local homology groups* at a point $x \in X$ are the groups $H_n(X, X \setminus \{x\})$.

(a) Let X be a space and let $x \in X$ be a point such that $\{x\}$ is closed in X. Use excision to show that, if U is any neighbourhood of x, then

$$H_n(X, X \setminus \{x\}) \cong H_n(U, U \setminus \{x\}).$$

Hence the groups only depend on the local topology of X near x.

(b) Explain why a homeomorphism $f : X \to Y$ must induce isomorphisms

$$f_* : H_n(X, X \setminus \{x\}) \xrightarrow{\cong} H_n(Y, Y \setminus \{f(x)\}).$$

Conclude that the local homology groups must be equal if two spaces are locally homeomorphic at a pair of points.

(c) Prove the following theorem.

Theorem (Invariance of dimension). Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be nonempty open subsets. If U and V are homeomorphic, then $m = n$.

Hint: Show $H_n(U, U \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$. Consider the LES of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$.

6. (a) Recall that a *tangent vector field* to the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a continuous map $v : S^n \to \mathbb{R}^{n+1}$ such that $v(x)$ is tangent to S^n at x, i.e., $v(x)$ is perpendicular to the vector x for each x. Let $v(x)$ be a nonvanishing tangent vector field on the sphere S^n. Show that

$$f_t(x) = \cos(\pi t)x + \sin(\pi t) \left(\frac{v(x)}{||v(x)||} \right)$$

is a homotopy from the identity map $id_{S_n} : S^n \to S^n$ to the antipodal map $-id_{S_n} : S^n \to S^n$.

(b) Prove the following theorem.

Theorem (Hairy ball theorem). The sphere S^n admits a nonvanishing continuous tangent vector field if and only if n is odd.

Remark: This result is alternately called the "Hedgehog Theorem" or the "'You can't comb a coconut' Theorem".