Terms and concepts covered: Categories; objects; morphisms; examples. Monic and epic morphisms. Covariant / contravariant functors. Universal property. Free groups: construction and universal property.

Corresponding reading: Hatcher Ch 0, "Cell complexes", "Homotopy extension property". Any reference on basic category theory (like Wikipedia or Tai-Danae Bradley's blog).

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Suppose that X is a topological space with equivalence relation $\sim$, and let $X/\sim$ be the quotient space with the quotient topology. Let $I = [0,1]$ be the unit interval with the Euclidean topology. Let
$$q : X \times I \to (X/\sim) \times I$$
be the natural map.
   (a) Show that q is a quotient map, i.e., a set U in $(X/\sim) \times I$ is open if and only if $q^{-1}(U)$ is open. *Hint:* You can quote Munkres ''Elements of algebraic topology", Theorem 20.1,

   Theorem. Let $p : X \to (X/\sim)$ be a quotient map, and let C be a locally compact Hausdorff space. Then
$$p \times id_C : X \times C \to (X/\sim) \times C$$
   is a quotient map.

   (Bonus) Prove Munkres Theorem 20.1.

   (b) Using the universal product of the quotient topology, explain why a homotopy $(X/\sim) \times I \to Y$ is continuous if and only if it arises from a continuous map $X \times I \to Z$ which is, for each fixed $t \in I$, constant on equivalence classes in X.

   This exercise shows how to verify continuity of homotopies on quotient spaces. You are welcome to asssume this result in our course (and you will not be tested on its proof).

2. Verify that $\partial(D^n \times D^m) = (\partial D^n \times D^m) \cup (D^n \times \partial D^m)$. Draw pictures for some small values of m, n.

3. Let X be a CW complex. Consider the interval I as a CW complex with two vertices and one edge.
   (a) Describe the natural CW complex structure on $X \times I$. What is its n-skeleton, in terms of the skeleta of X?
   (b) Show that the n-skeleton of $X \times I$ is contained in $X^n \times I$. Use this to deduce that a homotopy $X \times I \to Y$ is continuous if and only if its restriction to $X^n \times I$ is continuous for every n.

4. Draw a collection of finite graphs (in the sense of graph theory). In each graph G, identify a maximal tree T. Verify that the quotient G/T is a wedge of 1-spheres. Use Assignment Problem 2 to explain why the quotient map $G \to G/T$ is a homotopy equivalence.

5. **(Monic and epic morphisms).**
   (a) Consider the category of sets, the category of abelian groups, and the category of topological spaces. Prove that in these categories, a morphism is monic if and only if it is a injective map.
   (b) Consider the category of sets, the category of abelian groups, and the category of topological spaces. Prove that in these categories, a morphism is epic if and only if it is a surjective map.
   (c) Prove that in the category of rings, the map $\mathbb{Z} \to \mathbb{Q}$ is an epic morphism that is not surjective.

6. **Definition (Isomorphism).** Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is an *isomorphism* if there exists a morphism $g : Y \to X$ in $\mathscr{C}$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$. Then we write $g = f^{-1}$, and we say that the objects X and Y are *isomorphic*.

(a) Verify that this definition is agrees with your notion of "isomorphism" in every context you have encountered it.

(b) Recall that the *homotopy category* hTop is the category of topological spaces and homotopy classes of continuous maps. Verify that an isomorphism in this category is precisely a homotopy equivalence.

(c) Verify that "isomorphism" is an equivalence relation on objects in $\mathscr{C}$.

(d) Let $\mathscr{C}$ be a category containing objects A and B, and let F be a functor $F : \mathscr{C} \to \mathscr{D}$. Show that if A and B are isomorphic objects of $\mathscr{C}$, then $F(A)$ and $F(B)$ will be isomorphic objects of $\mathscr{D}$.

7. **(Groups as categories).** Given a group G, define a category $\mathscr{G}$ with a single object $\bigstar$ and morphisms $\mathrm{Hom}_{\mathscr{G}}(\bigstar, \bigstar) = \{g \mid g \in G\}$. The composition law is given by the group operation.

   (a) Show that a function between groups $G \to H$ is a group homomorphism if and only if the corresponding map between categories $\mathscr{G} \to \mathscr{H}$ is a functor.

   (b) (For those who have studied group representations). For a field k, let k–vect be the category of k-vector spaces and k-linear maps. Show that the definition of a functor from $\mathscr{G}$ to k–vect is equivalent to the definition of a linear representation of G over k.

8. Let Grp be the category of groups. Consider the map Z : Grp $\to$ Grp that takes every group G to itself and every morphism f to the zero map. Is Z a functor?

9. **(Power set functors).** Let f Set denote the category of finite sets and all functions between sets. Let $\mathscr{P}$: f Set $\to$ f Set be the function that takes a finite set A to its *power set* $\mathscr{P}(A)$, the set of all subsets of A. If $f : A \to B$ is a function of finite sets, let $\mathscr{P}(f) : \mathscr{P}(A) \to \mathscr{P}(B)$ be the function that takes a subset $U \subseteq A$ to the subset $f(U) \subseteq B$.

   (a) Show that $\mathscr{P}$ is a covariant functor.

   (b) What if we had instead defined $\mathscr{P}(f) : \mathscr{P}(B) \to \mathscr{P}(A)$ to take a subset $U \subseteq B$ to its preimage $f^{-1}(U) \subseteq A$ under f?

10. **(Open subsets functor).** Let Top be the category of topological spaces and continuous maps. Let Set be the category of sets and all functions of sets. Define a *contravariant* functor $\mathcal{O}$: Top $\to$ Set that takes a topological space X to its collection $\mathcal{O}(X)$ of open subsets. How should we define $\mathcal{O}$ on morphisms to make it well-defined and functorial?

11. **(More adjoints).** Let Top be the category of topological spaces and continuous maps. Let Set be the category of sets and all functions of sets. Let $\mathcal{F}$ be the "forgetful map"

$$\mathcal{F} : \underline{\text{Top}} \longrightarrow \underline{\text{Set}}$$

that takes a space X to its underlying set. Define maps

$$I, D : \underline{\text{Set}} \longrightarrow \underline{\text{Top}}$$

so that for a set S, $D(S)$ is the set S with the discrete topology, and $I(S)$ is the set S with the indiscrete topology. Prove that there are bijections

$$\mathrm{Hom}_{\underline{\text{Set}}}(A, \mathcal{F}(X)) \cong \mathrm{Hom}_{\underline{\text{Top}}}(D(A), X)$$

and

$$\mathrm{Hom}_{\underline{\text{Set}}}(\mathcal{F}(X), A) \cong \mathrm{Hom}_{\underline{\text{Top}}}(X, I(A)).$$

It turns out that these bijections are "natural", so this result shows that D is a *left adjoint* to $\mathcal{F}$, and I is the *right adjoint* to $\mathcal{F}$.

Assignment questions

(Hand these questions in!)

1. **(Real and complex projective space: cell structures).**

   Definition (Real projective space). *Real n-dimensional projective space* $\mathbb{R}\mathrm{P}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Specifically, $\mathbb{R}\mathrm{P}^n$ is the quotient space

   $$\mathbb{R}\mathrm{P}^n \quad = \quad (\mathbb{R}^{n+1} \setminus \{0\}) \quad / \quad x \sim \lambda x \qquad \text{for any } \lambda \in \mathbb{R} \setminus \{0\}$$

   Definition (Complex projective space). *Complex n-dimensional projective space* $\mathbb{C}\mathrm{P}^n$ is the space of 1-dimensional subspaces [1] through the origin in $\mathbb{C}^{n+1}$,

   $$\mathbb{C}\mathrm{P}^n \quad = \quad (\mathbb{C}^{n+1} \setminus \{0\}) \quad / \quad x \sim \lambda x \qquad \text{for any } \lambda \in \mathbb{C} \setminus \{0\}$$

   Read Hatcher Example 0.4 and 0.6 for a description of CW complex structures on $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$. Summarize their construction here. (You may read the book as you write up your solution.) Illustrate $\mathbb{R}\mathrm{P}^1$, $\mathbb{R}\mathrm{P}^2$, and $\mathbb{C}\mathrm{P}^1$ with pictures.

2. The goal of this question is to prove the following theorem.

   Theorem (Collapsing contractible subcomplexes). Let X be a CW complex and $A \subseteq X$ a subcomplex. If A is contractible, then the quotient $X \to X/A$ is a homotopy equivalence.

   As an intermediate step, we will prove another important result: the homotopy extension property for CW complexes. For this question, you may read Hatcher Chapter 0 "The Homotopy Extension Property" as you write up your solutions.

   (a) Define (qualitatively) a deformation retraction from $D^n \times I$ to $(D^n \times \{0\}) \cup (\partial D^n \times I)$.

   (b) Let X be a CW complex, and A a subcomplex. Hatcher writes,

   "This deformation retraction [from part (a)] gives rise to a deformation retraction of $X^n \times I$ onto $(X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I)$, since $X^n \times I$ is obtained from $(X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I)$ by attaching copies of $D^n \times I$ along $(D^n \times \{0\}) \cup (\partial D^n \times I)$."

   Explain this construction in the case that X is the CW complex structure on the 2-disk shown in Figure 1, and A is the left edge. Illustrate (with pictures) the deformation retraction from $X^0 \times I$, $X^1 \times I$, and $X^2 \times I$.

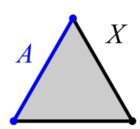
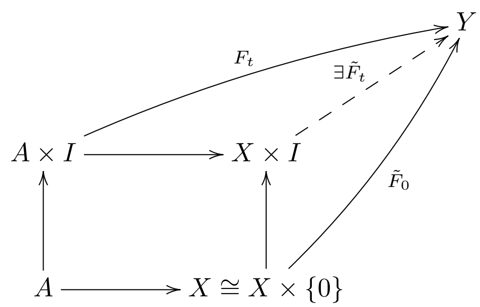
   

   Figure 1: A CW complex structure on D^2

   (c) Let A be a subcomplex of a CW complex X. Show that $(X \times \{0\}) \cup (A \times I)$ is a deformation retraction of $X \times I$. *Hint:* Perform the deformation retraction on $X^n \times I$ for the time interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. See Hatcher Proposition 0.16. You do not need to provide point-set details.

   (d) **Definition (Homotopy extension property).** Let X be a topological space and $A \subseteq X$ a subspace. We say that the pair (X, A) has the *homotopy extension property* if, given a homotopy $F_t(a)$ from $A \times I \to Y$ and a map $\tilde{F}_0 : X \to Y$ such that $\tilde{F}_0|_A = F_0$, then there is a homotopy $\tilde{F}_t(x)$ from $X \to Y$ such that $\tilde{F}_t|_A = F_t$. The homotopy$\tilde{F}_t(x)$ is called an *extension* of $F_t(a)$.

[1] 1-dimensional complex subspsaces are sometimes called *complex lines*, even though they are 2 real dimensionnal



The lift $\tilde{F}_t(x)$ need not be unique.

Let A be a subcomplex of a CW complex X. Show that (X, A) has the homotopy extension property. *Hint*: Use the pasting lemma, and the retraction defined by the deformation retraction from part (c) at time $t = 1$.

(e) Read Hatcher Proposition 0.17, which proves our theorem. Explain the steps in the proof and give explicit (if qualitative) descriptions of possibilities for the maps in the case that X is the graph in Figure 2 and A is its central edge.

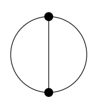
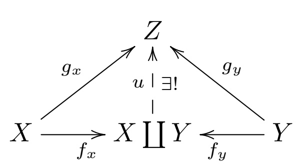


Figure 2: The theta graph

(f) Let X be a space and $A \subseteq X$ a subspace. The *cone* CA on A is the quotient space of $A \times I$ where $A \times \{1\}$ is collapsed to a point. Let Y be the space obtained by gluing CA to X by the identification $(a, 0) \sim a$ for all $a \in A$. Assuming that Y has a CW complex structure for which CA is a subcomplex, briefly explain why $Y \simeq X/A$.

(g) Our theorem does not hold for arbitrary contractible subspaces. Let $X = S^1$ and let A be the complement of a point in S^1, so A is homeomorphic to an open interval. Prove that S^1/A and S^1 are not homotopy equivalent, by proving S^1/A is contractible. (Next week we will prove that S^1 is not contractible).

3. **(Coproducts).** Let $\mathcal{C}$ be a category with objects X and Y. The *coproduct* of X and Y (if it exists) is an object $X \coprod Y$ in $\mathcal{C}$ with maps $f_x : X \to X \coprod Y$ and $f_y : Y \to X \coprod Y$ satisfying the following universal property: whenever there is an object Z with maps $g_x : X \to Z$ and $g_y : Y \to Z$, there exists a unique map $u : X \coprod Y \to Z$ that makes the following diagram commute:



(a) Let X and Y be objects in $\mathcal{C}$. Show that, if the coproduct $(X \coprod Y, f_x, f_y)$ exists in $\mathcal{C}$, then the universal property determines it uniquely up to unique isomorphism.

(b) Explain how to reinterpret this universal property as a bijection of sets

$$\mathrm{Hom}_{\mathcal{C}}(X \coprod Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, Z) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z)$$

for objects X, Y, Z.

(c) Prove that in the category of sets, the coproduct $X \coprod Y$ of sets X and Y is their disjoint union.

(d) Let Top be the category of topological spaces and continuous maps. The coproduct of $X \coprod Y$ of spaces X and Y is called the *(topological) disjoint union*. The underlying set is the disjoint union. Describe the topology on the disjoint union and check that it satisfies the universal property.

(e) Prove that in the category of abelian groups, the coproduct of groups $X \coprod Y$ is the direct sum $X \oplus Y$ with the canonical inclusions of X and Y. In other words, this universal property defines the direct sum operation on abelian groups.

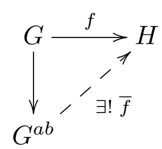
(f) In the category Grp of groups, the univeral property for the coproduct does *not* define the direct product operation. The coproduct $G \coprod H$ of groups G and H is a construction called the *free product* of G and H, and denoted $G * H$. Determine how to construct the group $G * H$ along with maps $G \to G * H$ and $H \to G * H$ that satisfy the universal property.

4. **(Abelianization).** Let Grp denote the category of groups and group homomorphisms, and let Ab denote the category of abelian groups and group homomorphisms. Define the *abelianization* G^{ab} of a group G to be the quotient of G by its *commutator subgroup* $[G, G]$, the subgroup normally generated by *commutators*, elements of the form $ghg^{-1}h^{-1}$ for all $g, h \in G$.

(a) Define a map of categories $[-,-] :$ Grp $\to$ Grp that takes a group G to its commutator subgroup $[G, G]$, and a group morphism $f : G \to H$ to its restriction to $[G, G]$. Check that this map is well defined (ie, check that $f([G, G]) \subseteq [H, H]$) and verify that $[-,-]$ is a functor.

(b) Show that G^{ab} is an abelian group. Show moreover that if G is abelian, then $G = G^{ab}$.

(c) Show that the quotient map $G \to G^{ab}$ satisfies the following universal property: Given any **abelian** group H and group homomorphism $f : G \to H$, there is a unique group homomorphism $\overline{f} : G^{ab} \to H$ that makes the following diagram commute:



This universal property shows that G^{ab} is in a sense the "largest" abelian quotient of G.

(d) Show that the map ab that takes a group G to its abelianization G^{ab} can be made into a functor $ab :$ Grp $\to$ Ab by explaining where it maps morphisms of groups $f : G \to H$, and verifying that it is functorial.

(e) The category Ab is a subcategory of Grp. Define the functor $\mathcal{A} :$ Ab $\to$ Grp to be the inclusion of this subcategory; $\mathcal{A}$ takes abelian groups and group homomorphisms in Ab to the same abelian groups and the same group homomorphisms in Grp. Briefly explain why the universal property in Part (c) can be rephrased as follows: Given groups $G \in$ Grp and $H \in$ Ab, there is a natural bijection between the sets of morphisms:

$$\mathrm{Hom}_{\underline{\mathrm{Grp}}}(G, \mathcal{A}(H)) \cong \mathrm{Hom}_{\underline{\mathrm{Ab}}}(G^{ab}, H)$$

Remark: Since this bijection is "natural" (a condition we won't formally define or check) it means that $\mathcal{A} :$ Ab $\to$ Grp and $ab :$ Grp $\to$ Ab are what we call a pair of *adjoint functors*.