Terms and concepts covered: Covering spaces, lifting properties of covering spaces.

Corresponding reading: Hatcher Ch 1.3

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

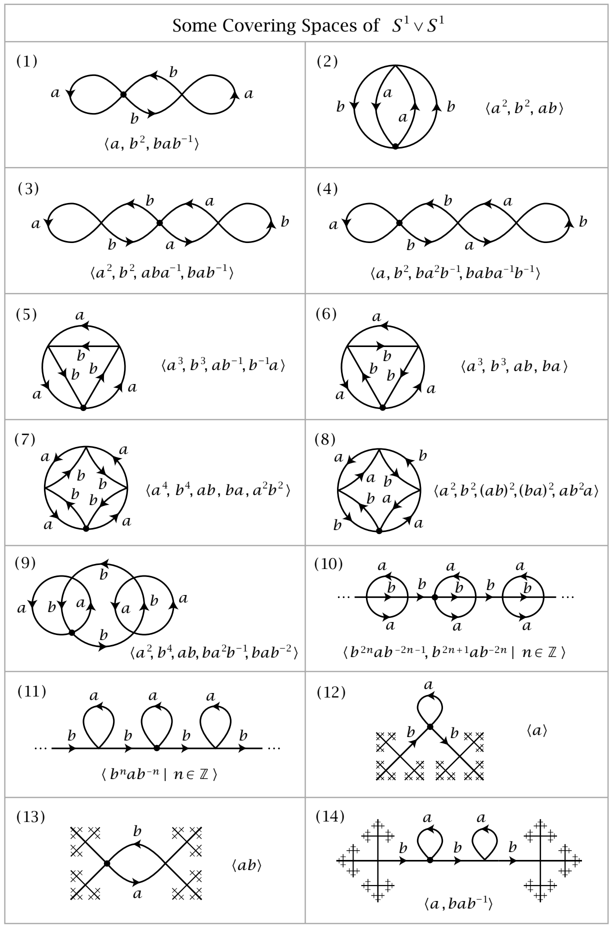
1. Suppose that X is a connected space. Show that the only subsets of X that are both open and closed are X and $\varnothing$.
2. (a) Let $p : \tilde{X} \to X$ be a map and suppose an open subset $U \subseteq X$ is evenly covered by p (in the sense of Assignment Problem 1). Show that any open subset $V \subset U$ is also evenly covered by p.
   (b) Let $p : \tilde{X} \to X$ be a covering space. Deduce that every point of X has a neighbourhood basis of evenly covered open sets.
3. Let $p : \tilde{X} \to X$ be a covering map.
   (a) Let A be a subspace of X. Show that $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is a covering map.
   (b) Suppose B is a subspace of $\tilde{X}$. Is $p|_B : B \to p(B)$ necessarily a covering map?
4. Let $p : \tilde{X} \to X$ be a covering map. For $x_0 \in X$, show that $p^{-1}(x_0)$ is a discrete set.
5. Some but not all sources require a covering map $p : \tilde{X} \to X$ to be surjective. Show that, even if p is not surjective, its image must be a union of connected components of X. *Hint:* Assignment Problem 2.
6. Prove that a covering map $p : \tilde{X} \to X$ is an open map, i.e., the image of an open subset is open.
7. **Definition (local homeomorphism).** A continuous map $f : X \to Y$ is a *local homeomorphism* if every point $x \in X$ has a neighbourhood U such that $f(U) \subseteq Y$ is open, and the restriction $f|_U : U \to f(U)$ is a homeomorphism.

   Definition (locally homeomorphic). A space X is *locally homeomorphic* to a space Y if every point in X has an open neighbourhood homeomorphic to an open subset of Y.

   Note that this definition is not symmetric in X and Y.

   (a) Show that if there exists a local homeomorphism $X \to Y$, then X is locally homeomorphic to Y. The converse is not true, for example, S^2 and $\mathbb{R}^2$ are locally homeomorphic to each other, but no local homeomorphism exists $S^2 \to \mathbb{R}^2$.
   (b) Verify that a covering map $p : \tilde{X} \to X$ is a local homeomorphism.
   (c) Verify that a local homeomorphism $f : X \to Y$ preserves local properties. For example, X will satisfy each of the following properties if and only if $f(X)$ does.
      (i) local connectedness and local path-connectedness
      (ii) local compactness
      (iii) first countability (every point has a countable neighbourhood basis)
      (iv) being locally Euclidean
8. Find an example of a continuous map that is a local homeomorphism but not a covering map.
9. Let $p : \tilde{X} \to X$ be a covering map. Show that, if X is Hausdorff, then so is $\tilde{X}$.
10. Consider the covers $p : \tilde{X} \to S^1 \vee S^1$ shown on Hatcher p58 (copied below).
    (a) For each cover, verify that it is a cover and that $p_*(\pi_1(\tilde{X}))$ is the subgroup shown.

(b) Consider the automorphism group of directed, labelled graphs $\tilde{X}$. This means the graph automorphisms that preserve the labels a and b and their orientations. For each cover $\tilde{X}$, determine whether this automorphism group acts transitively on vertices of $\tilde{X}$.



11. Show that S^n is a cover of $\mathbb{R}\mathrm{P}^n$.

12. Prove that a 1-sheeted cover is precisely a homeomorphism.

13. (a) For each n, construct an n-sheeted cover of S^1.
    (b) Show that there is no 3-sheeted cover of $\mathbb{R}\mathrm{P}^2$.
    Hint: Assignment Problem 2. What are the subgroups of $\pi_1(\mathbb{R}\mathrm{P}^2)$?

14. See Assignment Problem 4 for the definitions of simply-connected, locally simply-connected, and semi-locally simply-connected.
    (a) Show that a simply-connected space is semi-locally simply-connected.
    (b) Show that a locally simply-connected space is semi-locally simply-connected.
    (c) Verify that S^1 is locally simply-connected and semi-locally simply-connected but not simply-connected.
    (d) Verify that the infinite earring is path-connected and locally path-connected but not semi-locally simply-connected.
    (e) Let CI be the cone on the infinite earring (in the sense of Homework #2 Problem 2(f)). Show that CI is simply-connected and semi-locally simply-connected, but not locally simply-connected.

(f) Show that the topological disjoint union $CI \sqcup CI$ is semi-locally simply connected, but not simply connected or locally simply-connected.

15. Let $p : S^1 \to S^1$ be the cover $e^{i\theta} \mapsto e^{3i\theta}$ that "wraps" the circle around 3 times. Choose a basepoint x_0 in the base space and a lift $\tilde{x_0}$.
    (a) Show that the map p_* induced on fundamental groups is the map
    $$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Z} \\ 1 &\longmapsto 3. \end{aligned}$$
    (b) Explicitly describe for this cover, with pictures, the map Φ from Assignment Problem 2. Verify that the map is well-defined on the cosets of the subgroup $H = 3\mathbb{Z}$ in $\mathbb{Z}$, and bijective.

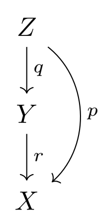
16. (a) Explain how we can identify the universal cover of S^1 constructed in 4 with our cover $\mathbb{R} \to S^1$.
    (b) Describe a cover of S^1 associated to each subgroup of $\mathbb{Z}$.

17. (a) Explain how we can identify the universal cover of the torus constructed in 4 with $\mathbb{R}^2$.
    (b) Explain how to construct a cover of the torus from $\mathbb{R}^2$ for any subgroup of $\mathbb{Z}^2$.

Assignment questions

(Hand these questions in!)

1. Let p, q, r be continuous maps with $p = r \circ q$. Assume X is locally connected. Show that, if p and r are covering maps, then so is q.



We will use this result later in our proof of the classification of covering spaces of a topological space.

Hint: The following terminology may be convenient.[1] Let $p : \tilde{X} \to X$ be a continuous map of spaces. An open subset $U \subseteq X$ is called *evenly covered* by p if $p^{-1}(U)$ is the disjoint union of open subsets $\sqcup V_i \subseteq \tilde{X}$ such that, for each i, the restriction $p|_{V_i} : V_i \to U$ is a homeomorphism. We call the parts V_i of the partition $\sqcup V_i$ of $p^{-1}(U)$ *slices*. With this terminology, p is a covering map if and only if every point $x \in X$ has a neighbourhood which is evenly covered.

2. (a) Recall that a function N on a space X is *locally constant* if each $x \in X$ has a neighbourhood where N is constant. Show that a locally constant function is in fact constant on connected components of X.
   (b) **Definition (Sheets of a cover).** Let X be a connected space. The number of *sheets* of a cover $p : \tilde{X} \to X$ is the cardinality of $p^{-1}(x)$ for a point $x \in X$.
   Verify that the cardinality of $p^{-1}(x)$ is locally constant on X, and deduce that the number of sheets is well-defined for a cover of a connected space X.

[1] I do not think it is universally standard but it is used by Munkres.

(c) Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a cover with $X, \tilde{X}$ path-connected. Consider the function

$$\begin{aligned} \phi : \pi_1(X, x_0) &\longrightarrow p^{-1}(x_0) \\ [\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

where $\tilde{\gamma}$ is a lift of a representative γ starting at $\tilde{x_0}$. Show ϕ is well-defined.

(d) Show moreover that ϕ is well-defined on right cosets of $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, and so defines a function

$$\begin{aligned} \Phi : \pi_1(X, x_0) \text{ mod } H &\longrightarrow p^{-1}(x_0) \\ H[\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

(e) Show that Φ is bijective. This proves the following theorem.

Theorem (Sheets of a cover and π_1). Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a cover with $X, \tilde{X}$ path-connected. The number of sheets of p is equal to the index of $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ in $\pi_1(X, x_0)$.

(f) The group $\mathbb{Z}^2$ has index-4 subgroups $(4\mathbb{Z} \times \mathbb{Z})$, $(\mathbb{Z} \times 4\mathbb{Z})$ and $(2\mathbb{Z} \times 2\mathbb{Z})$. Find a 4-sheeted covering map of the torus corresponding to each. No justification necessary.

3. (a) Suppose $p : \tilde{X} \to X$ is a covering map and that $\tilde{X}$ is compact. Show that p is finite-sheeted (on each component of X).

(b) Let $p : \tilde{X} \to X$ be a surjective covering map that is finite-sheeted (on each component of X). Show that X is compact if and only if $\tilde{X}$ is.

4. **(Construction of the universal cover).** Throughout this question, when we refer to "path homotopy", or "homotopy classes of paths", we implicitly mean homotopy rel $\{0, 1\}$.
Hint: You may read Hatcher p 63-65 while you complete this question.

Definition (simply-connected). A space X is called 0-*connected* if it is path-connected. The space is *simply-connected* or 1-*connected* if it is path-connected and $\pi_1(X) = 0$.

Definition (Locally simply-connected). A space X is *locally simply-connected* if each point $x \in X$ has a neighbourhood basis of simply-connected open sets U.

Definition (Semi-locally simply-connected). A space X is called *semi-locally simply-connected* if every point $x \in X$ has a neighbourhood U such that the inclusion $U \hookrightarrow X$ induces the trivial map $\pi_1(U, x) \to \pi_1(X, x)$.

Observe that in the definition of semi-locally simply-connected, the neighbourhood U does not need to be simply-connected. A loop in U based at x may not contract to the constant map at x by a path homotopy in U, but it does contract to the constant map by a path homotopy in the larger space X. See Warm-Up Problem 14.

Fact: CW complexes are locally contractible, therefore locally simply-connected and semi-locally simply-connected. Hatcher proves this in Proposition A.4 and in this course you may assume it without proof.

The goal of this problem is to construct a simply-connected cover of any path-connected, locally path-connected, semi-locally simply-connected space. We will later show that such a cover is (in an appropriate sense) unique, and it is called the *universal cover*.

(a) Suppose that X is simply-connected. Show that any two points in X are joined by a unique homotopy class of paths.

(b) Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering map. Use the lifting properties to show that there is a bijection between homotopy classes of paths in X starting in x_0 and homotopy classes of paths in $\tilde{X}$ starting at $\tilde{x_0}$.

(c) **Definition (The universal cover of X).** Let X be a path-connected, locally path-connected, semi-locally simply-connected space with basepoint x_0. Define

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ a path in } X \text{ based at } x_0\}$$

where $[\gamma]$ is the homotopy class of the path γ. Define

$$\begin{aligned} p : \tilde{X} &\longrightarrow X \\ [\gamma] &\longmapsto \gamma(1) \end{aligned}$$

Let $\mathcal{U} = \{U \subseteq X \mid U \text{ is path-connected, open, and } \pi_1(U) \to \pi_1(X) \text{ is trivial}\}$. We topologize $\tilde{X}$ be defining a basis of open sets

$$U_{[\gamma]} = \{[\gamma{\cdot}\alpha] \mid \; U \in \mathcal{U}, \;\; \gamma \text{ a path from } x_0 \text{ to a point in } U, \;\; \alpha \text{ a path in } U \text{ with } \alpha(0) = \gamma(1)\;\}.$$

Show that the map p is well-defined and surjective.

(d) Show that $\mathcal{U}$ is a basis for the topology on X.

(e) Show that $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$. Conclude that the sets $U_{[\gamma]}$ form the basis for a topology on $\tilde{X}$.

(f) Show that p is continuous. *Hint:* Show that, for $U \in \mathcal{U}$,

$$p^{-1}(U) = \bigcup_{\gamma \text{ a path from } x_0 \text{ to } U} U_{[\gamma]}.$$

(g) Show that $p|_{U_{[\gamma]}}$ is a homeomorphism to U. Deduce that p is a covering map.

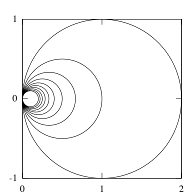
(h) Show that $\tilde{X}$ is path-connected. *Hint:* For a point $[\gamma] \in \tilde{X}$, consider the path $t \mapsto [\gamma_t]$, where $[\gamma_t] \in \tilde{X}$ is the homotopy class of the path

$$\gamma_t(s) = \begin{cases} \gamma(s), & 0 \leq s \leq t \\ \text{constant function at } \gamma(t), & t \leq s \leq 1. \end{cases}$$

(i) Let $[x_0]$ denote the class of the constant path in X at x_0. Show that $\pi_1(\tilde{X}, [x_0]) = 0$.
Hint: We will prove in class that p_* is injective, so it suffices to show that its image is trivial. For a loop γ in X based at x_0, first show that $t \to [\gamma_t]$ is a lift of γ. Deduce that this lift is not a loop unless γ is trivial in $\pi_1(X, x_0)$.

(j) Give a brief/informal explanation of how we can identify the universal cover of $S^1 \vee S^1$ with the infinite tree shown in Figure 1.

(k) **(QR Exam, January 2022).** For $n \in \mathbb{N}$, let C_n be the metric circle of radius $\frac{1}{n}$ in $\mathbb{R}^2$ with its north pole at the origin $(0, 0)$. Let C be the union $\bigcup_n C_n$, topologized as a subspace of Euclidean 2-space. The space C has been called the *infinite earring*, the *Hawaiian earring*, and the *shrinking wedge of circles*.



The space C is a standard example of a space that is not *semi-locally simply connected*. Prove that C does not have a universal cover by verifying that it is not semi-locally simply connected, and proving that every space with a universal cover is semi-locally simply connected.

Edit: For the purposes of part (k), take the definition of a universal cover of a path-connected, locally path-connected space X to be a covering space $p : \tilde{X} \to X$ such that $\tilde{X}$ is simply connected. We will soon prove that, if it exists, there is a unique such cover (up to a suitable notion of isomorphism of covers) and so it must agree with the covering space you constructed above in this Assignment Problem.

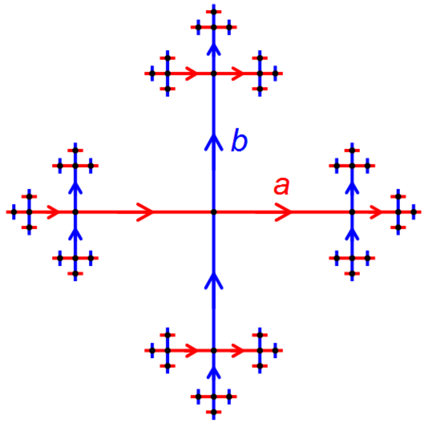


Figure 1: The universal cover of $S^1 \vee S^1$

5. **(The covering space of X associated to $H \subseteq \pi_1(X)$).** Let X be a path-connected, locally path-connected, semi-locally simply-connected space with basepoint x_0. Let $H \subseteq \pi_1(X, x_0)$ be a subgroup. Define X_H to be the quotient of the universal cover $\tilde{X}$ of X (Assignment Problem 4) by the equivalence relation

$$[\gamma] \sim [\gamma'] \quad \text{iff} \quad \gamma(1) = \gamma'(1) \text{ and } [\gamma \cdot \overline{\gamma'}] \in H.$$

Hint: You may read Hatcher Prop 1.36 while you complete this question.

(a) Verify that $\sim$ is well-defined and is an equivalence relation.

(b) Show that the covering map $p : \tilde{X} \to X$ factors through a map $p_H : X_H \to X$.

(c) Verify that p_H is a covering map.

(d) Show that the image of $(p_H)_*$ is H.