Terms and concepts covered: lifting properties of covering spaces, classification of covering spaces

Corresponding reading: Hatcher Ch 1.3

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. **(Point-set review).** Let $f : X \to Y$ be a map of topological spaces. Verify that the following are equivalent.
   - f is continuous.
   - The preimage $f^{-1}(U)$ is open for every open subset $U \subseteq Y$.
   - The preimage $f^{-1}(C)$ is closed for every closed subset $C \subseteq Y$.
   - For every $x \in X$ and neighbourhood U of $f(x)$, there exists a neighbourhood V of x such that $f(V) \subseteq U$. [Note: If this condition holds for a particular point x, we say that f is *contitnuous at x*.]
   - Given a choice of (sub)basis for the topology on Y, $f^{-1}(B)$ is open for every (sub)basis element B.
   - For every subset $A \subseteq X$ there is containment $f(\overline{A}) \subseteq \overline{f(A)}$
   - (If X is first-countable, e.g. a metric space or a CW complex) f is sequentially continuous: If $(a_n)_{n \in \mathbb{N}}$ is a sequence of points in X converging to some limit a_∞, then $\big(f(a_n)\big)_{n \in \mathbb{N}}$ converges and its limit is $f(a_\infty)$.

2. (a) Give an example of a space that is simply connected but not contractible [we'll have the tools to prove this later in the course].
   (b) Prove that a graph is simply connected if and only if it is contractible (that is, if it is a tree).

3. A *torsion* group G is a group where every element has finite order. Show that the only group homomorphism from a torsion group G to a free abelian group $\mathbb{Z}^n$ is the zero map. What if G is generated by torsion elements?

4. Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be covering maps, and let $f : X_1 \to X_2$ be an isomorphism of covers (Assignment Problem 3 (a)).
   (a) Show that, for each $x \in X$, the map f defines a bijection between $p_1^{-1}(x)$ and $p_2^{-1}(x)$.
   (b) Show that f^{-1} is also an isomorphism of covers.
   (c) Verify that "isomorphism of covers" defines an equivalence relation on the covers of a fixed space X.
   (d) Fix a cover $p : \tilde{X} \to X$. Show that the isomorphisms $\tilde{X} \to \tilde{X}$ form a group under composition.
   (e) Compute the group of isomorphisms for the following covers.
   - $\mathbb{R} \to S^1$
   - The N-sheeted cover $S^1 \to S^1$
   - $S^n \to \mathbb{RP}^n$
   - Your favourite covers from Hatcher's table on p58 (shown below).

5. Let $p : \tilde{X} \to X$ be the covering map of a connected cover, and let $\tau : \tilde{X} \to \tilde{X}$ be a deck transformation. Prove that if τ fixes a point, then τ is the identity map.

6. Let H be a subgroup of a group G.
   (a) Define the *normalizer* $N_G(H)$ of H in G.
   (b) Show that $N_G(H)$ is a subgroup of G.

(c) Show that H is contained in $N_G(H)$, and is normal in $N_G(H)$.

(d) Show that if H is a normal subgroup of G, then $N_G(H) = G$.

(e) Show that $N_G(H)$ is maximal in the following sense: if J is a subgroup $H \subseteq J \subseteq G$ and H is normal in J, then $J \subseteq N_G(H)$.

Assignment questions

(Hand these questions in!)

1. (a) The following lemma is proved in Hatcher Lemma 1A.3.

   Lemma. Let X be a graph. Then every cover $\tilde{X}$ of X is a graph, with vertices and edges the lifts of vertices and edges, respectively, in X.

   Describe how to define a 1-dimensional CW complex structure on a cover $\tilde{X}$ of a graph X. You do not need to give point-set details. You may read Hatcher while you write your solution.

   (b) Prove the following theorem.

   Theorem. Every subgroup of a free group is free.

2. (a) **(Topology QR Exam, May 2016).** Let X be the complement of a point in the torus $S^1 \times S^1$.

   (i) Compute $\pi_1(X)$.

   (ii) Show that every map $\mathbb{RP}^n \to X$ is nullhomotopic for $n \geq 2$.

   (b) **(Topology QR Exam, Jan 2022).** Let G be a graph, that is, a 1-dimensional CW complex. Let S^2 denote the 2-sphere. For each of the following statements, either prove the statement, or give (with justification) a counterexample.

   (i) Every continuous map $G \to S^2$ is nullhomotopic.

   (ii) Every continuous map $S^2 \to G$ is nullhomotopic.

   Some hints [which were not given on the QR exam]: (i) Cellular approximation theorem, (ii) you can assume the result of Warm-Up Problem 2 without proof.

3. **(The Classification of Covering Spaces).**

   (a) **Definition (Isomorphism of covers).** Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be covering maps. A continuous map $f : X_1 \to X_2$ is an *isomorphism of covers* if f is a homeomorphism and $p_1 = p_2 \circ f$.

   Use the lifting properties and uniqueness of lifts proved in class to prove the following proposition.

   Proposition (Uniqueness of the cover associated to a subgroup of $\pi_1(X)$). If X is path-connected and locally path-connected, then two path-connected covering spaces $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ are isomorphic via an isomorphism $f : X_1 \to X_2$ taking a basepoint $\tilde{x_1} \in p_1^{-1}(x_0)$ to a basepoint $\tilde{x_2} \in p_2^{-1}(x_0)$ if and only if

   $$(p_1)_*(\pi_1(X_1, \tilde{x_1})) = (p_2)_*(\pi_1(X_2, \tilde{x_2})).$$

   (b) Deduce the following important theorem, which is the culmination of your work on this and the previous assignment.

   Theorem (The classification of (based) covering spaces). Let X be path-connected, locally path-connected, and semi-locally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ to the covering space $(\tilde{X}, \tilde{x_0})$.

(c) Let $p : \tilde{X} \to X$ be a covering map with $\tilde{X}$ path-connected. Let $x_0 \in X$ and let $\tilde{x_0}, \tilde{x_1} \in p^{-1}(x_0)$. Analyze the change-of-basepoint map on $\tilde{X}$ to prove that $p_*(\pi_1(X, \tilde{x_0}))$ and $p_*(\pi_1(X, \tilde{x_1}))$ are conjugate subgroups of $\pi_1(X, x_0)$.

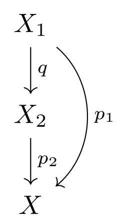
(d) Prove the following variation on the classification theorem.

Theorem (The classification of (unbased) covering spaces). Let X be path-connected, locally path-connected, and semi-locally simply-connected. Then there is a bijection between the set of isomorphism classes of path-connected covering spaces $p : \tilde{X} \to X$ and the set of conjugacy classes of subgroups of $\pi_1(X)$.

4. **(The Galois correspondence for covering spaces).** In this question, we assume all spaces are path-connected, locally path-connected, and semi-locally simply-connected.

(a) Suppose that H_1, H_2 are subgroups of the fundamental group $\pi_1(X, x_0)$ of a space X, and let $p_1 : (X_1, x_1) \to (X, x_0)$ and $p_2 : (X_2, x_2) \to (X, x_0)$ be the covering spaces such that $(p_1)_*$ and $(p_2)_*$ induce the inclusions of H_1 and H_2, respectively, into $\pi_1(X, x_0)$.

Explain why p_1 factors through p_2 (as in the diagram below) if and only if $H_1 \subseteq H_2$.



Conclude from Homework 5 Assignment Problem #1 that, if it factors, the map q is a covering map.

(b) Give a precise statement of the resulting strengthening of our classification theorem for based covering spaces of X: for every subgroup of $\pi_1(X, x_0)$ there is a unique covering space, and for every inclusion of subgroups $H_1 \to H_2$ there is an intermediate covering map.

Remark: There is, in fact, an isomorphism of posets between the subgroups of $\pi_1(X)$ (ordered by inclusion) and the covers of X (ordered by existence of intermediate covers). For a more detailed statement, see Hatcher Chapter 1.3 Problem 24. This result is sometimes called the *Galois correspondence* for covering spaces, in analogy to the Galois correspondence for field extensions.

(c) Let $X = \mathbb{RP}^2 \times \mathbb{RP}^2$. Draw the diagram of based covering maps of X and intermediate covers, and label the fundamental group of each space.

5. **(The action of $\pi_1(X, x_0)$ on the fibres).** Let X be a connected, locally path-connected, semi-locally simply-connected space. Let $p : \tilde{X} \to X$ be a covering map, and let α be a path in X. Define a map

$$L_\alpha : p^{-1}(\alpha(0)) \to p^{-1}(\alpha(1))$$

as follows: for a point $\tilde{x_0} \in p^{-1}(\alpha(0))$, lift α to the path $\tilde{\alpha}$ starting at $\tilde{x_0}$. Then $L_\alpha(\tilde{x_0}) = \tilde{\alpha}(1)$.

(a) Explain why L_α only depends on the homotopy class of α rel $\{0, 1\}$.

(b) Show that L_α is a bijection of sets. *Hint:* What is its inverse?

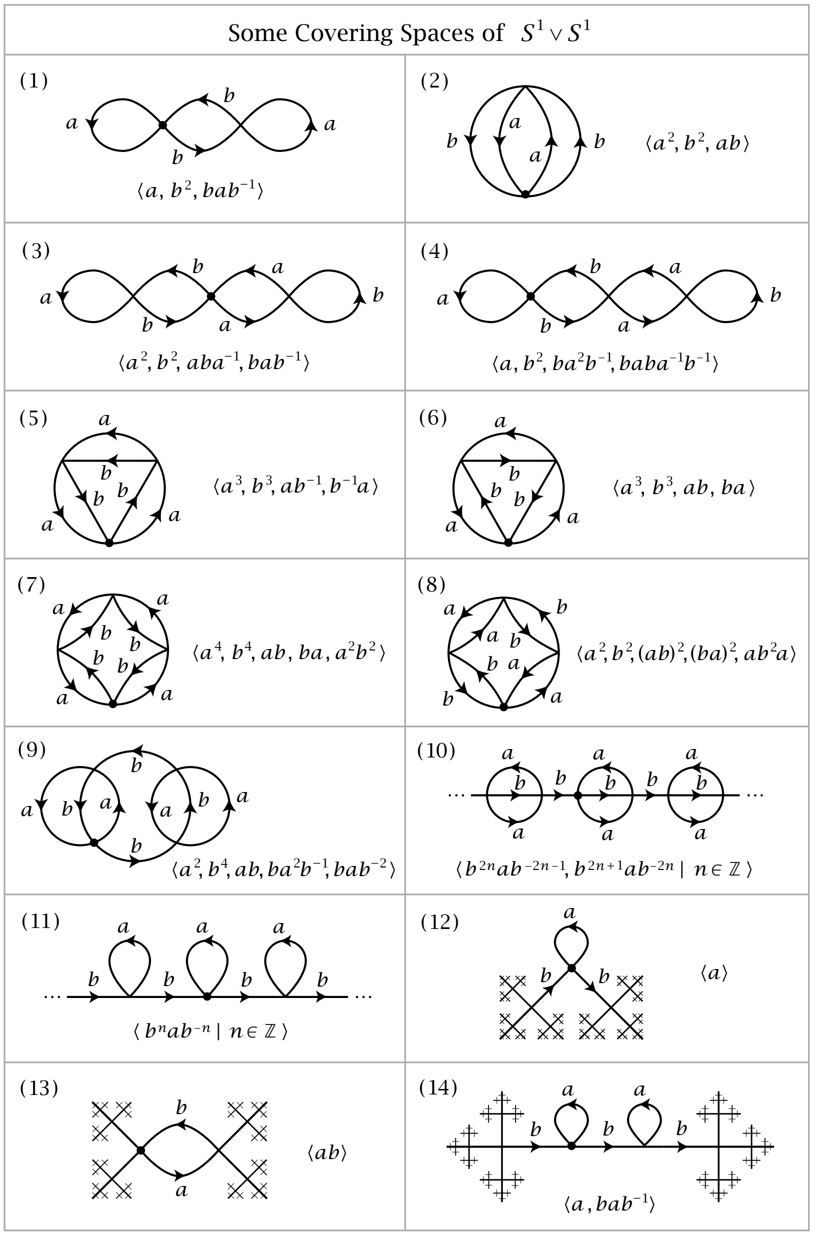
(c) Show that $L_{\overline{\alpha \cdot \beta}} = L_{\overline{\alpha}} \circ L_{\overline{\beta}}$.
(Note that we had to replace α by its inverse $\overline{\alpha}$ to make this relationship covariant).

(d) Now let us restrict to classes $[\gamma] \in \pi_1(X, x_0)$. Conclude that the assignment

$$\begin{aligned} \pi_1(X, x_0) &\longrightarrow \{\text{Permutations of } p^{-1}(x_0)\} \\ [\gamma] &\longmapsto L_{\overline{\gamma}} \end{aligned}$$

defines a group action of $\pi_1(X, x_0)$ on the set $p^{-1}(x_0)$.

(e) Choose five covers $\tilde{X}$ of $S^1 \vee S^1$ from Hatcher's table from p58 (copied below). Describe the permutation on the vertices of $\tilde{X}$ defined by the generator a, and the permutation defined by the generator b. No justification necessary; just state your answer.



(f) Recall the map Φ defined in Homework 5 Assignment Problem 2(d)

$$\begin{aligned} \Phi : \pi_1(X, x_0) \text{ mod } H &\longrightarrow p^{-1}(x_0) \\ H[\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

that defined a bijection between $p^{-1}(x_0)$ and the right cosets of $H = p_*(\pi_1(\tilde{X}, \tilde{x_0})$. Show that the group action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ defined above corresponds to the usual action of $\pi_1(X, x_0)$ on the right cosets by right multiplication,

$$[\gamma] \cdot (H[\beta]) = H[\beta \cdot \overline{\gamma}]$$

(g) Deduce that, if H is normal in $\pi_1(X, x_0)$, the action of $\pi_1(X, x_0)$ induces a well-defined action by the quotient group $\pi_1(X, x_0)/H$.

In fact, we can reconstruct the cover $p : \tilde{X} \to X$ from the action of $\pi_1(X, x_0)$ on the fibre $F = p^{-1}(x_0)$ by taking a suitable quotient of $\tilde{X_0} \times F$, where $\tilde{X_0}$ is the universal cover. (This construction is described on p69-70 of Hatcher). Hatcher concludes that the n-sheeted covers of X are classifed by conjugacy classes of group homomorphisms from $\pi_1(X, x_0)$ to the symmetric grroup S_n.

6. (a) Let X be a wedge of n circles, so $\pi_1(X, x_0) = F_n$. Let $h : F_n \to G$ be a surjective group homomorphism. Explain how we could use the results of Assignment Problems 5 and 1 (a) to construct the graph $\tilde{X}$ covering X with fundamental group the subgroup ker$(h) \subseteq \pi_1(X, x_0)$. Explain moreover how we can use the cover $\tilde{X}$ to determine a free generating set for ker(h).

   (b) **(Topology QR Exam, May 2017).** Let F be the free group on a, b. Let $G = \{1, x, x^2\}$ be the cyclic group on three generators written multiplicatively. Let $h : F \to G$ be a homomorphism which sends $a \mapsto x, b \mapsto x^2$. Find free generators of Ker(h).

   (c) **(Topology QR Exam, Jan 2017).** Let F be the free group on a, b. Let G be a symmetric group (=group of all permutations) on three elements, and let $x, y \in G$ be elements of order 2 and 3, respectively. Let $h : F \to G$ be a homomorphism which sends $a \mapsto x, b \mapsto y$. Find free generators of Ker(h).