Terms and concepts covered: Deck transformations, regular covers.

Corresponding reading: Hatcher Ch 1.3

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let X be path-connected, locally path-connected semi-locally simply-connected space. Explain why, if X is simply connected, the only covers of X are homeomorphisms $X \to X$.
2. Let X, Y be path-connected, locally path-connected spaces. Assume Y is semi-locally simply connected. Given a map $f : (X, x_0) \to (Y, y_0)$, which path-connected covers $\tilde{Y}$ of Y will f lift to?
3. We have illustrated a cover $\tilde{X}$ of $S^1 \vee S^1$ as a graph where every edge is directed and labelled by either a or b. Explain how this convention encodes the data of the covering map, and explain why a deck transformation of $\tilde{X}$ is precisely a graph automorphism that preserves the direction and label of every edge.
4. (a) Suppose that X has an abelian fundamental group. Explain why every connected cover of X is regular.
   (b) Explain why every connected 2-sheeted cover is regular.
5. Let G be a group acting on a set X.
   (a) Define *orbit*. Prove that the condition "x_1 and x_2 are in the same orbit" defines an equivalence relation on X.
   (b) State and prove the orbit-stabilizer theorem.
6. **Definition (Group acting on a space, I).** Let G be a group. A group action of G on a space Y is a group homomorphism $\rho : G \to \text{Homeo}(Y)$, where $\text{Homeo}(Y)$ is the group of homeomorhpisms $Y \to Y$.

   Note: If G were a topological group, we would want to impose extra conditions on our group action to be compatible with the topology. Here we assume that G has no topology, or, equivalently, we may assume that G has the discrete topology.

   Verify that the definition of a group action on a space is equivalent to the following.

   Definition (Group acting on a space, II). Let G be a group. A group action of G on a space Y is a map

   $$\begin{aligned} \alpha : G \times Y &\longrightarrow Y \\ (g, y) &\longmapsto g \cdot y \end{aligned}$$

   satisfying three conditions.

   (i) For each fixed $g \in G$, the coresponding map is continuous:

   $$\begin{aligned} g : Y &\longrightarrow Y \\ y &\longmapsto g \cdot y \end{aligned}$$

   (ii) For each $g, h \in G$ and $y \in Y$, $(gh) \cdot y = g \cdot (h \cdot y)$.
   (iii) For $e \in G$ the identity, $e \cdot y = y$ for all $y \in Y$.
7. (a) Let G be a group with a covering space action on a space Y (Assignment Problem 2). Prove that the action is free.
   (b) Show by example that a free action of a group G on a space Y need not be a covering action. *Hint:* $\mathbb{R}$ acts on $\mathbb{R}$.
8. Let $p : Z \to W$ be a continuous surjective map. Show that, if p is an open map, then it is a quotient map.

Assignment questions

(Hand these questions in!)

1. (a) Let F_n be the free group on n generators, and F_m the free group on m generators. Show that, if $F_m \cong F_n$, the $m = n$. Conclude that the number n (called the *rank* of F_n) is an isomorphism invariant. *Hint:* abelianization & structure theorem for finitely generated abelian groups

   (b) Let X be a connected, finite graph with n vertices and e edges. Show that $\pi_1(X)$ is the free group of rank $(e - n + 1)$. You may assume the following result from graph theory.

   Proposition (Combinatorics of trees). Let T be a finite tree (which is by definition connected). If T has n vertices, then it has $(n - 1)$ edges.

   (c) Let X be a connected m-sheeted cover of the wedge $\bigvee_n S^1$ of n circles. We proved on Homework 6 that $\pi_1(X)$ is a free group. What is the rank of $\pi_1(X)$?

   (d) Prove the following theorem.

   Theorem (Schreier index formula). Let F_n be the free group of rank n. A subgroup of index $m \in \mathbb{N}$ in F_n has rank $1 + m(n - 1)$. An infinite-rank subgroup has infinite index.

   This theorem shows that the rank of a finite-index subgroup is a function of its index. Moreover, the larger the index (so "smaller" the subgroup in F_n), the larger its rank!

   (e) Show by example that an infinte-index subgroup of F_n can have finite or infinite rank. *Hint:* See Hatcher's table of covering spaces.

   (f) **(QR Exam, Aug 2021).** Let F_n denote the free group on n letters $\{a, b, c, \dots\}$.

   (i) Prove that F_4 does **not** have a finite-index subgroup isomorphic to F_8.

   (ii) Construct a finite-index subgroup H of F_4 isomorphic to F_7. Determine (explaining your steps) a free generating set for H, and explain whether H is normal.

2. **(Covering spaces as quotients by covering actions).** You may refer to Hatcher p71-73 while you write your solution to this problem.

   (a) **Definition (Orbit space).** Let G be a group acting on a space Y. Recall that the *orbit* of a point $y \in Y$ is the subset
   $$G \cdot y = \{g \cdot y \mid g \in G\} \subseteq Y.$$
   The *orbit space* of this action, denoted Y/G, is the quotient space of Y in which every orbit is collapsed to a point.

   Let $p : \tilde{X} \to X$ be a surjective normal covering map with deck group $G(\tilde{X})$. Verify that we can identify X with the orbit space $\tilde{X}/G(\tilde{X})$, and $p : \tilde{X} \to X$ with the quotient map. *Hint:* Warm-up problem 8.

   (b) **Definition (Covering space action).** Let G be a group acting on a space Y. Then this action is *covering space action* if it satisfies the following condition. Each $y \in Y$ has a neighbourhood U such that all images $g(U)$ for distinct $g \in G$ are disjoint. In other words, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$.

   Let $p : \tilde{X} \to X$ be a connected covering space. Verify that the action of the deck group $G(\tilde{X})$ is a covering space action.

   (c) Now suppose that a group G is acting on a space Y by a covering space action. Prove that the quotient map $p : Y \to Y/G$ is a normal covering space.

   (d) Let G is acting on a space Y by a covering space action, and suppose Y is path-connected. Prove that the Deck group of the cover $p : Y \to Y/G$ is isomorphic to G.

   (e) Let G is acting on a space Y by a covering space action, and suppose Y is path-connected and locally path-connected. Let $p : Y \to Y/G$ be the quotient. Prove that
   $$\frac{\pi_1(Y/G, G \cdot y_0)}{p_*(\pi_1(Y, y_0))} \cong G.$$
   In particular, if Y is simply-connected, then $\pi_1(Y/G) \cong G$. *Hint:* This part is a result from class.