Terms and concepts covered: n-simplex; vertices, subsimplices, and faces; boundary and interior of a simplex; Δ-complex; chain complex, n-chains, exactness, homology groups; boundary homomorphisms, cycles and boundaries, simplicial homology groups, homology classes.

Corresponding reading: Hatcher Chapter 2, Introduction and Section 2.1, "Δ-Complexes" and "Simplicial Homology".

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. **Definition (Convex).** A subset C in Euclidean space is *convex* if it contains the line segment connecting any pair of its points.

   Definition (Convex combination, convex hull). Let X be a subset of Euclidean space. A *convex combination* of points in X is a sum of the form

   $$t_1x_1 + t_2x_2 + \cdots t_nx_n \qquad \text{such that } x_i \in X, t_i \in \mathbb{R}, t_i \geq 0, t_1 + \cdots + t_n = 1.$$

   The *convex hull* of X is the set of all convex combinations of points in X.

   (a) Prove that the convex hull of X is the minimal (under inclusion) convex subset containing X.

   (b) Prove that the convex hull of X is the intersection of all convex subsets containing X.

2. Let $\Delta^n = [v_0, v_1, \ldots, v_n]$ be the standard n-simplex,

   $$\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; t_i \geq 0, \sum_i t_i = 1 \right\}.$$

   What are its vertices? Show that the convex hull of any $(k+1)$ of its vertices is canonically homeomorphic to a k-simplex. Conclude that it therefore makes sense topologically (as well as combinatorially) to call this subspace a k-dimensional subsimplex.

3. Let $\Delta^n = [v_0, v_1, \ldots, v_n]$ be an n-dimensional simplex. For each $k \leq n$, how many k-dimensional subsimplices does Δ^n have?

4. Describe the canonical Δ-complex structure on an n-simplex. What is its k-skeleton?

5. In this question, we will find another way to coordinatize an n-simplex. Let

   $$\Delta^0_* = \{0\},$$

   $$\Delta^n_* = \{(s_1, \ldots, s_n) \in \mathbb{R}^n \mid 0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq 1\}$$

   (a) Draw Δ^n_* for $n = 0, 1, 2, 3$.

   (b) Recall that we defined the standard simplex $\Delta^n = \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1\}$. Show that Δ^n is homeomorphic to Δ^n_* via the map

   $$s_i = t_0 + t_1 + \cdots + t_{i-1}.$$

6. Consider our coordinatization of the n-simplex,

   $$\Delta^n_* = \{(s_1, \ldots, s_n) \in \mathbb{R}^n \mid 0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq 1\}.$$

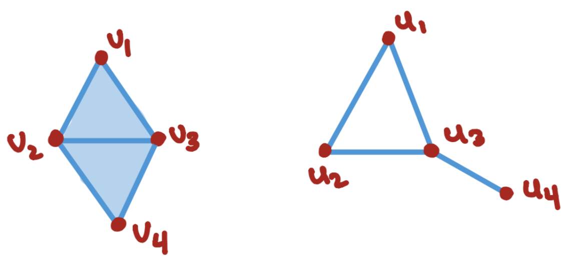
   Prove that the boundary $\partial\Delta^n$ and the open simplex $\mathring{\Delta}^n$ are indeed the boundary and interior of Δ^n, respectively, in the usual sense of point-set topology, when Δ^n is viewed as the subset Δ^n_* of $\mathbb{R}^n$.

7. (a) Verify that an n-simplex (as a topological space) is homeomorphic to a closed n-ball.
   (b) Verify that a Δ-complex structure on a space X is, in particular, a CW complex structure.

8. (a) Which of our standard CW complex structures on the spheres S^1 and S^2 are Δ-complex structures?
   (b) Is our standard CW complex structure on $\mathbb{R}\mathrm{P}^2$ a Δ-complex structure?

9. Choose your preferred name for the ∂ symbol.

10. Let X be either of the Δ-complexes shown below. Let $C_n(X)$ denote the associated nth simplicial chain group, and let $\partial_n : C_n(X) \to C_{n-1}(X)$ be the boundary map.



   (a) We choose the total orderings of the vertices v_1, v_2, v_3, v_4 and u_1, u_2, u_3, u_4, respectively, for the two complexes. Explain how this determines an order on the vertices of every simplex. Label the edges of each complex with the appropriate direction.
   (b) Compute the boundary (that is, the image under ∂_n) of the following n-chains.
      (i) $2[v_1, v_2, v_3]$
      (ii) $[v_1, v_2, v_3] + [v_2, v_3, v_4]$
      (iii) $[v_1, v_2, v_3] - [v_2, v_3, v_4]$
      (iv) $[u_1, u_2] - [u_1, u_3] + [u_2, u_3]$
      (v) $[u_1, u_2] - [u_1, u_3] + [u_2, u_3] + [u_3, u_4]$
   (c) Explain for each calculation how this boundary relates to your intuitive geometric understanding of "boundary".

11. Let X be a Δ-complex. Let $C_n(X)$ denote the nth simplicial chain group, and let $\partial_n : C_n(X) \to C_{n-1}(X)$ be the boundary map.
   (a) Verify that $\partial_n \circ \partial_{n+1} = 0$.
   (b) Give a geometric interpretation of the equation $\partial_n \circ \partial_{n+1} = 0$, in the spirit of "a boundary has no boundary".

12. (a) Let u, v be vertices in a simplicial complex joined by an edge. What is the relationship between the (oriented) edge $[u, v]$ and the (oriented) edge $[v, u]$? What is the relationship between the 1-chains $[u, v]$ and $-[v, u]$, and the relationships between their boundaries?
   (b) Let $\Delta_n = [v_0, v_1, \ldots, v_n]$ be an n-simplex, and τ a permutation in S_{n+1}. Show that

$$\partial_n([v_{\tau(0)}, v_{\tau(1)}, \ldots, v_{\tau(n)}]) = \begin{cases} \partial_n([v_0, v_1, \ldots, v_n]) & \text{if } \tau \text{ is an even permutation} \\ -\partial_n([v_0, v_1, \ldots, v_n]) & \text{if } \tau \text{ is an odd permutation.} \end{cases}$$

   Conclude that our ordering of the vertices does matter in our computation of the differential—different orders result in different signs—but order does not matter up to even permutations.

13. Let (C_*, d_*) be a chain complex, and suppose it is exact at every point C_n. Such sequences are called *exact sequences*. What is the homology of (C_*, d_*)?

14. Let (C_*, d_*) be a chain complex supported in degree n, that is,

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow C_n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0.$$

What is the homology of (C_*, d_*)?

15. Let (C_*, d_*) be a chain complex.
    (a) Suppose that the differential d_n is identically zero for some n.

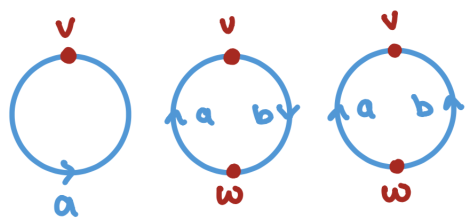
    $$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n=0} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

    Show that $H_{n-1} = \ker(d_{n-1})$, and $H_n = C_n/\mathrm{im}(d_{n+1})$.
    (b) Suppose the differential d_n is identically zero for every n. Show that $H_n = C_n$ for every n.

16. Compute the simplicial homology of the disjoint union of n points.

17. Compute the simplicial homology of a 1-simplex.

18. Compute the simplicial homology of S^1 with each of the following Δ-complex structures, with the given orientations of the edges.



19. Compute the simplicial homology groups of the wedge $\bigvee_k S^1$ of k circles.

20. Let X be a Δ-complex, and $C_n(X)$ its n^{th} simplicial chain group.
    (a) Show that $C_0(X) = \ker(\partial_0)$, so $C_0(X)$ is the group of 0-cycles. Conclude that, topologically, a 0-cycle is any linear combination of vertices of X.
    (b) Show that two vertices in $C_0(X)$ are homologous exactly if they are connected via a path of edges in X.
    (c) Conclude that $H_0(X)$ consists of formal sums of equivalence classes of vertices of X, where two vertices are equivalent if they are in the same path-component of X.
    (d) Explain the sense in which $H_0(X)$ "is" the free abelian group on the path components of X.

21. (a) Let (C_*, d_*) be a chain complex. Explain why, if the n^{th} homology group H_n has rank N, then the n^{th} chain group C_n must have had rank at least N.
    (b) Let X be a space. We will show that the homology groups are homeomorphism invariants (in fact, homotopy invariants). Explain why, if if the n^{th} simplicial homology group $H_n(X)$ has rank N, then any Δ-complex structure on X must have at least N simplices of dimension n.

22. Review the structure theorem for finitely generated abelian groups.

Assignment questions

(Hand these questions in!)

1. **(Covering spaces as quotients by covering actions, ctd).** This is a continuation of Assignment Problem 2 from Homework #7. You may refer to Hatcher p72-73 while you write your solution to this problem.
   (a) On Homework 5 Problem 2(f), you constructed the covers of the torus associated to the subgroups $4\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 4\mathbb{Z}$, and $2\mathbb{Z} \times 2\mathbb{Z}$ of its fundamental group $\mathbb{Z}^2$. Explain how you could construct these covering spaces using a suitable action of the groups $4\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 4\mathbb{Z}$, and $2\mathbb{Z} \times 2\mathbb{Z}$, respectively, on the universal cover $\mathbb{R}^2$ of the torus.
   (b) Suppose we have a covering space action of a group G on a simply connected space Y. Let $H_1 \subseteq H_2 \subseteq G$ be subgroups. Explain how to use the action of H_1, H_2 on Y to construct the intermediate cover $q : X_1 \to X_2$ defined in Homework 6, Assignment Problem #4. What happens in the special cases $H_1 = 0$ or $H_2 = G$? You do not need to check details.

2. **(Lens space).**
   (a) Prove that any free action of a finite group on a Hausdorff space Y is a covering space action.
   (b) Let $S^3 \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$ be the unit sphere. For coprime integers p, q, define an action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 by

   $$(z_1, z_2) \longmapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2).$$

   Verify that this action is free.
   (c) The orbit space $S^3 \,/\, \mathbb{Z}/p\mathbb{Z}$ is a 3-manifold called a *lens space*. What is its fundamental group? *Hint:* Assignment Problem 2 from Homework #7.

3. (a) Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $f \circ g \simeq id_Y$ and $h \circ f \simeq id_X$.
   (b) Let X and $\tilde{X}$ be path-connected and locally path-connected. Let $p : \widetilde{X} \to X$ be a regular covering map, and let $\widetilde{f}$ be a map making the following diagram commute.

   $$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ \big\downarrow p & & \big\downarrow p \\ X & \xrightarrow{id_X} & X \end{array}$$

   Prove that $\widetilde{f}$ must be a deck transformation, that is, verify that $\widetilde{f}$ is a homeomorphism.
   (c) Let X and Y be path-connected, locally path-connected, semi-locally simply connected spaces, and assume they are homotopy-equivalent. Prove that their universal covers are homotopy equivalent.

4. For each of the following spaces, define a Δ-complex structure on the space, and compute its simplicial homology groups.
   (a) a 2-simplex
   (b) S^2
   (c) a torus
   (d) a Mobius band

5. **Definition (Morphism of chain complexes).** A *morphism f_* of chain complexes* or *chain map* from (C_*, ∂_*) to (D_*, δ_*) is a sequence of group homomorphisms $f_n : C_n \to D_n$ making the following diagram commute.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow{\delta_{n+1}} & D_n & \xrightarrow{\delta_n} & D_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots \end{array}$$

(a) Verify that a morphism f_* of chain complexes induces well-defined group homomorphisms on the homology groups

$$f_n : H_n(C_*) \to H_n(D_*)$$

for every n.

(b) **Definition (Quasi-isomorphism).** A morphism of chain complexes $f_* : (C_*, \partial_*) \to (D_*, \delta_*)$ is a *quasi-isomorphism* if the maps induced on homology are all isomorphisms.

Give an example of a quasi-isomorphism of chain complexes where at least one map f_n is not an isomorphism.

6. **(Homomorphisms of free abelian groups).** Let A be an $n \times n$ integer matrix, viewed as $\mathbb{Z}$-linear map from $\mathbb{Z}^n$ to $\mathbb{Z}^n$.

(a) Suppose that A has rank n. Prove that the kernel of A is trivial.
(Note: Here we mean 'rank' in the usual sense from linear algebra, for example, it is the rank of A when A is viewed as a matrix with entries in $\mathbb{Q}$).

(b) Show by example that, even if A has rank n, it need not be surjective.

(c) The *cokernel* of a map of abelian groups is the quotient of its codomain by its image. Prove or find a counterexample: if the map A has rank n, then the cokernel of A must be finite.

(d) Prove the $\mathbb{Z}$-module version of the rank-nullity theorem: If A is an $(m \times n)$ matrix of rank k, then its kernel is a free abelian subgroup of $\mathbb{Z}^n$ of rank $(n - k)$.