Terms and concepts covered: singular n-chains, singular homology groups. Induced maps on homology. Chain homotopy. Reduced homology groups. Good pair, long exact sequence of a pair.

Corresponding reading: Hatcher Ch 2.1, "Singular homology", "Homotopy invariance", "Exact sequences and excision" to end of page 114, Ch 2.A "Homology and fundamental group".

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Show that, if a 2-simplex T in a Δ-complex is glued along the word in the edges $a_1a_2a_3$, then $\partial T = a_1 + a_2 + a_3$.

2. (a) Given a chain complex,

$$\ldots \xrightarrow{d_{n+2}} C_{n+1}(X) \xrightarrow{d_{n+1}} C_n(X) \xrightarrow{d_n} C_{n-1}(X) \xrightarrow{d_{n-1}} \ldots$$

   explain why the homology group $H_n(X)$ depends only on the groups $C_{n+1}(X)$, $C_n(X)$, $C_{n-1}(X)$, and the maps d_{n+1} and d_n.

   (b) Let X be a Δ-complex. We proved that a generating set for $\pi_1(X)$ is determined by its 1-skeleton X^1, and that the relations for $\pi_1(X)$ (and hence the isomorphism type) are determined by the 2-skeleton X^2.
   Let $H_n(X)$ be the nth simplicial homology group of X. Explain the sense in which generators for $H_n(X)$ (cycles) are determined by the n-skeleton X^n, and relations for $H_n(X)$ (boundaries) are determined by the $(n+1)$-skeleton X^{n+1}.

3. Let X be a space with a choice of Δ-complex structure. Explain the difference between the definitions of the simplicial n-chains on X, and the singular n-chains on X.

4. Let X be a space. Let $C_n(X)$ denote the singular n-chains on X, and let $H_n(X)$ denote the nth singular homology group. Suppose that X has path components $\{X_\alpha\}$.
   (a) Why must the image of each singular n-chain be contained in a single path-component X_α?
   (b) Fix n. Deduce that, as a group, $C_n(X)$ decomposes as a direct sum $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$.
   (c) Verify that the boundary map ∂_n respects this decomposition.
   (d) Conclude that there is a decomposition $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$

5. Let X be a point. Working directly from the definition of singular homology, show that

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \geq 1. \end{cases}$$

6. (a) Let X be a path-connected space, and let $H_n(X)$ denote its nth singular homology group. Working directly from the definition of singular homology, show that $H_0(X) \cong \mathbb{Z}$.
   (b) Let X be a space with path-components $\{X_\alpha\}_\alpha$. Use part (a) and Warm-up Problem 4 to show that $H_0(X) \cong \bigoplus_\alpha \mathbb{Z}$.

7. Let $f : X \to Y$ be a continuous map of topological spaces, and let $f_\#$ denote the map induced by f on singular n-chains,

$$\begin{aligned} f_\# : C_n(X) &\longrightarrow C_n(Y) \\ [\sigma : \Delta^n \to X] &\longmapsto [f \circ \sigma : \Delta^n \to Y]. \end{aligned}$$

(a) Verify that $f_\# \circ \partial = \partial \circ f_\#$.

(b) Conclude that $f_\#$ is a chain map, so for each n, there is an induced group homomorphism $f_* : H_n(X) \to H_n(Y)$.

8. Fix n. For a continuous map $f : X \to Y$ of topological spaces, let $f_* : H_n(X) \to H_n(Y)$ denote the induced map on singular homology groups, as in Warm-up Problem 7.

   (a) For maps of spaces $g : X \to Y$ and $f : Y \to Z$, verify that $(f \circ g)_* = f_* \circ g_*$.

   (b) Verify that $id_X : X \to X$ induces the identity map on $H_n(X)$.

   (c) Conclude that H_n is a functor from the category of topological spaces and continuous maps, to the category of abelian groups and group homomorphisms.

9. Let $f : X \to Y$ be a continuous map of path-connected spaces. Show that the induced map $f_* : H_0(X) \to H_0(Y)$ is an isomorphism.

10. Let $iA \subseteq X$, and let ι be the inclusion map. Show that, if A is a retract of X, then the induced map $\iota_* : H_n(A) \to H_n(X)$ is injective for all n.

11. We sketched a proof in class of the following result.

    Theorem (homotopic maps induce the same map on H_n). If $f, g : X \to Y$ are homotopic maps, then they induce the same map $f_* = g_*$ on singular homology groups.

    Show that this theorem (and functoriality of H_n) implies the following.

    Theorem (H_n is a homotopy invariant). Let $f : X \to Y$ be a homotopy equivalence. Then the induced map on singular homology $f_* : H_n(X) \to H_n(Y)$ is an isomorphism. In particular, homotopy equivalent spaces have isomorphic homology groups.

12. Let $f : X \to Y$ be a nullhomotopic map. Show that the induced map $f_* : H_n(X) \to H_n(Y)$ is zero for all $n \geq 1$, and that the induced map $f_* : \widetilde{H}_0(X) \to \widetilde{H}_0(Y)$ is zero. What is the induced map $f_* : H_0(X) \to H_0(Y)$?

13. **(Interpreting exact sequences).** Prove that ...

    (a) the sequence
    $$0 \longrightarrow A \xrightarrow{f} B$$
    is exact if and only if f is injective.

    (b) the sequence
    $$B \xrightarrow{g} C \longrightarrow 0$$
    is exact if and only if g is surjective.

    (c) the sequence
    $$0 \longrightarrow A \xrightarrow{h} B \longrightarrow 0$$
    is exact if and only if h is an isomorphism.

    (d) the sequence
    $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
    is exact if and only if f is injective, g is surjective, and $C \cong B/f(A)$, where $f(A) \cong A$.

14. **(Calculations with exact sequences of abelian groups).** The following sequences are exact.

    (a) Compute the group A. *Hint:* Which maps must be injective, surjective, zero?
    $$\ldots \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/5\mathbb{Z} \longrightarrow A \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \ldots$$

(b) Compute the group B.

$$\ldots 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow B \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/5\mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$$

(c) What are the possibilities for the group C?

$$\ldots \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow C \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow \ldots$$

15. **(Short Five Lemma).**

(a) Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\varphi'"'] & B' \arrow[r, "\psi'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

Prove the remaining step in the Short Five Lemma: If α and γ both surject, then β must also surject. Conclude that if α and γ are isomorphisms, then β must be an isomorphism.

(b) Explain why the following commutative diagram with exact rows does not contradict the short five lemma, even though $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ are not isomorphic.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\varphi"] \arrow[d, "\cong"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\psi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\varphi'"'] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r, "\psi'"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

16. Compute the singular homology groups and the reduced singular homology groups of the space X when X is the empty set.

17. Which of the following pairs of spaces $A \subseteq X$ are good pairs?

(a) $(M, \{p\})$ for M a manifold and $p \in M$.
(b) $(\mathbb{Q}, A)$ for A a proper closed subset
(c) $(X, \{0\})$ where $X = \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\} \subseteq \mathbb{R}$
(d) $(S^1, S^1 \setminus \{p\})$ for $p \in S^1$.
(e) (X, X^k), X a CW complex with k-skeleton X^k
(f) $(D^2, \partial D^2)$
(g) $(D^2, D^2 \setminus \partial D^2)$

18. Consider the maps $m : S^1 \to T$ and $\ell : S^1 \to T$ that are the inclusions of the meridian $S^1 \times \{1\}$ and longitudinal circle $\{1\} \times S^1$, respectively. See Assignment Problem 4. Explain how the induced maps $H_1(S^1) \to H_1(T)$ give a topological interpretation for the homology classes in $H_1(T)$. In general, we can sometimes understand degree-n homology classes in X in terms of the induced maps from a closed n-manifold.

19. Let A be a square matrix with entries in a commutative unital ring R. Recall that A is *invertible over* R if it has a 2-sided inverse matrix with entries in R.

(a) Suppose that A has entries in $\mathbb{Z}$, so we may view A as a matrix over $\mathbb{Z}$ or over $\mathbb{Q}$. Show by example that A may be invertible over $\mathbb{Q}$ but not over $\mathbb{Z}$. Explain why, if A is invertible over $\mathbb{Z}$, it is necessarily invertible over $\mathbb{Q}$.

(b) Show that A is invertible over R if and only if its determinant is a unit in R. In particular, a matrix with entries in $\mathbb{Z}$ is invertible over $\mathbb{Z}$ if and only if it has determinant ± 1.

(c) Explain why a matrix with entries in a field k is invertible over k if and only if it is invertible over any field extension of k.

Assignment questions

(Hand these questions in!)

1. Let A be a square matrix with entries in a commutative unital ring R. Recall we say A is *invertible over R* if it has a 2-sided inverse matrix with entries in R. See Warm-up Problem 19.

   Definition / Theorem (Smith normal form). Let A be an $m \times n$ matrix over a principal ideal domain R. There exists an $m \times m$ matrix S and an $n \times n$ matrix T such that S and T are invertible over R, and

$$SAT = \begin{bmatrix} \alpha_1 & 0 & 0 & & \cdots & & 0 \\ 0 & \alpha_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & \alpha_r & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{bmatrix}$$

   where the diagonal entries α_i satisfy $\alpha_i | \alpha_{i+1}$ for all $1 \leq i \leq r$. The matrix A is called the *Smith normal form* of A. The elements α_i are unique up to multiplication by a unit in R. They are called the *invariant factors* of A.

   We are interested in the case $R = \mathbb{Z}$.
   Note that, since S, T are invertible, the rank of A is equal to the rank of its Smith normal form.

   (a) Let A be a $\mathbb{Z}$-linear map $\mathbb{Z}^n \to \mathbb{Z}^m$ with invariant factors $\alpha_1, \alpha_2, \ldots \alpha_r$. Prove that the cokernel of A is isomorphic to $\mathbb{Z}^{m-r} \oplus \bigoplus_i \mathbb{Z}/\alpha_i\mathbb{Z}$. Conclude that Smith normal form can therefore be used to put a quotient of a free abelian group $\mathbb{Z}^m$ into standard form (standard in the sense of the structure theorem for finitely generated abelian groups), by writing generators for the kernel as the columns of a matrix.

   Remark: In fact, any proof of the structure theorem is likely implicitly a proof of existence/uniqueness of Smith normal form.

   (b) An integer matrix can be put in Smith normal form using the following row and column operations, which are invertible over $\mathbb{Z}$.

   R1. swap rows R_i and row R_j
   R2. multiply row R_i by -1
   R3. replace row R_i by $R_i + nR_j$ for some row $R_j \neq R_i$ and $n \in \mathbb{Z}$

   C1. swap columns C_i and row C_j
   C2. multiply column C_i by -1
   C3. replace column C_i by $C_i + nC_j$ for some row $C_j \neq C_i$ and $n \in \mathbb{Z}$

   To transform A into its Smith normal form, we use the following general steps. You may (if you wish) read a detailed description in the following handout

   `https://www3.nd.edu/~sevens/smithform.pdf`

   .

   - Let d be the gcd of all entries of A. Use row and column operations, and the Euclidean algorithm, to transform the matrix so that some matrix entry equal to d.

     Remark: Observe that the row and column operations do not change the gcd.
   - Use row and column swaps (R1 and C1) to place d in entry $(1, 1)$.

- Use row and column operations R3 and C3 to clear the first row and first column, to obtain a matrix of the form
$$\begin{bmatrix} d & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix}.$$
- Repeat the procedure on the matrix A'.

  Remark: Each row operation corresponds to multiplying A on the left by an invertible integer *elementary matrix*. Each column operation corresponds to multiplying A on the right by an invertible integer *elementary matrix*. Thus, by keeping track of the sequence of row and column operations applied, we can determine the matrices S and T as products of elementary matrices.

Explain and illustrate the steps to transform the following matrix into its Smith normal form.
$$A = \begin{bmatrix} 4 & 6 & 6 \\ 8 & 4 & 12 \end{bmatrix}$$
(You do not need to compute S and T). Verify your answer by going to the website

`https://sagecell.sagemath.org/`

and entering the lines

```
A = matrix([[4, 6, 6],[8, 4, 12]])
A.smith_form()
```

When you hit "Evaluate", SAGE will give you three matrices: the Smith normal form of A, and the matrices T and S.

(c) Let A be an $m \times n$ integer matrix, and let B be an $\ell \times m$ integer matrix, such that $BA = 0$.
$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "A"] \arrow[rr, bend right, "0"'] & \mathbb{Z}^m \arrow[r, "B"] & \mathbb{Z}^\ell \end{tikzcd}$$
Prove that B factors through a $\mathbb{Z}$-linear map $\overline{B} : \mathbb{Z}^m/\mathrm{im}(A) \to \mathbb{Z}^\ell$, and that
$$\ker(\overline{B}) = \ker(B)/\mathrm{im}(A).$$

(d) Prove the following.

Theorem (Smith normal form and homology computations). Let A be an $m \times n$ integer matrix, and let B be an $\ell \times m$ integer matrix, such that $BA = 0$.
$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "A"] \arrow[rr, bend right, "0"'] & \mathbb{Z}^m \arrow[r, "B"] & \mathbb{Z}^\ell \end{tikzcd}$$
Then
$$\ker(B)/\mathrm{im}(A) = \mathbb{Z}^{m-r-s} \oplus \bigoplus_{i=1}^{r} \mathbb{Z}/\alpha_i\mathbb{Z}$$
where $r = \mathrm{rank}(A)$, $s = \mathrm{rank}(B)$, and $\alpha_1, \ldots, \alpha_r$ are the invariant factors of A.

(e) Use part (d) and SAGE to compute the homology of the following chain complex.
$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} -30 & -54 \\ -16 & -55 \\ 3 & 9 \\ -2 & 7 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 41 & -90 & -178 & -162 \\ 34 & -74 & -144 & -134 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow 0$$

2. **Definition (Reduced homology).** Let X be a space, and let $C_n(X)$ denote its nth singular homology group. Define the *augmented* singular chain complex

$$\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$.

The *reduced* singular homology groups $\widetilde{H}_n(X)$ of X are the homology groups of this chain complex.

(a) Let C be a free abelian group with $\mathbb{Z}$-basis B, and let $\epsilon : C \to \mathbb{Z}$ be the homomorphism mapping every basis element to 1. The kernel K of ϵ is called the *augmentation ideal*. Show that K is generated by the elements $a - b$ for $a, b \in B$, and show that, given a distinguished element $b_0 \in B$, the set $\{b - b_0 \mid b \in B, b \neq b_0\}$ is a $\mathbb{Z}$-basis for K.

(b) Verify that the augmented singular chain complex is, in fact, a chain complex.

(c) Suppose $X = \varnothing$. Verify that

$$H_n(\varnothing) = 0 \quad \text{for all } n, \quad \text{and} \quad \widetilde{H}_n(\varnothing) = \begin{cases} 0, n \neq -1 \\ \mathbb{Z}, n = -1 \end{cases}$$

(d) Suppose $X \neq \varnothing$. Prove that

$$H_n(X) = \widetilde{H}_n(X), \ n \neq 0$$
$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$$

In particular, the singular homology groups and reduced singular homology groups only differ mildly in degree zero! Nevertheless, the reduced homology have some favourable combinatorial properties that make them often more convenient to work with. One reason is the following.

(e) Let X be a contractible space. Show that $\widetilde{H}_n(X) = 0$ for all n.

Remark: The reduced homology groups $\widetilde{H}_n$ define functors from Top to Ab.

3. In this problem, we will begin a proof of the following theorem.

Theorem ($H_1(X) \cong \pi_1(X, x_0)^{ab}$). Let X be path-connected space with basepoint x_0. There is a surjective group homomorphism

$$h : \pi_1(X, x_0) \longrightarrow H_1(X)$$
$$[\gamma] \longmapsto \text{singular 1-chain } \gamma$$

whose kernel is the commutator subgroup of $\pi_1(X, x_0)$. In particular,

$$H_1(X) \cong \pi_1(X, x_0)^{ab}.$$

You may read Hatcher 2.A and other relevant sections while you write your solutions.

(a) Let α be a based loop in (X, x_0). Explain how α is a singular 1-chain, and verify that α is a cycle.

(b) Suppose α is the constant loop at x_0. Show that α is a boundary, specifically, the boundary of the constant singular 2-simplex at x_0.

(c) Show that, if $\alpha \simeq \beta$ are homotopic rel $\{0, 1\}$, then α and β are homologous. *Hint:* Subdivide $I \times I$.

(d) If α and β are paths with $\alpha(1) = \beta(0)$, then the 1-chain $\alpha \cdot \beta$ is homologous to the 1-chain $\alpha + \beta$. In particular the result applies to based loops α, β.
Hint: Define a singular 2-simplex with boundary α, β, and $\alpha \cdot \beta$.

(e) If α is a based path and $\overline{\alpha}$ its inverse, show that the 1-chain $\overline{\alpha}$ is homologous to $-\alpha$.

(f) **(h is a homomorphism).** Deduce that h is a well-defined homomorphism. It is a special case of the *Hurewicz homomorphism*.

(g) **(h is surjective).** Let $x \in H_1(X)$, and let $\sum_i n_i \sigma_i$ be a 1-cycle representing x. By allowing repeats of summands σ_i, we can assume each coefficient n_i is ± 1. Show that x is in the image of h. You may use these steps:

- Explain why we may assume each n_i is 1.
- Explain why we may assume each σ_i is a loop, by inductively replacing sums $\sigma_i + \sigma_j$ with the product of paths $\sigma_i \cdot \sigma_j$.
- Explain why we can assume σ_i is a loop based at x_0, possibly by replacing σ_i by a homotopic loop of the form $\eta_i \cdot \sigma_i \cdot \eta_i^{-1}$.
- Find $[\gamma] \in \pi_1(X, x_0)$ such that $h([\gamma]) = x$.

(h) **($[\pi_1, \pi_1] \subseteq \ker(h)$).** Explain why the commutator subgroup of $\pi_1(X, x_0)$ must be contained in the kernel of h.

Next week, we will complete this problem with a proof that $\ker(h) \subseteq [\pi_1, \pi_1]$.

4. (a) Let X be a Δ-complex, and let $f : Y \to X$ be the inclusion of a Δ-subcomplex. Show that f induces a well-defined homomorphism on reduced simplicial homology groups, $f_* : \widetilde{H}_*(Y) \to \widetilde{H}_*(X)$.

   (b) Compute the maps induced on reduced homology by the following maps of topological spaces. *Hint:* For some of these maps, you can solve the problem by viewing their homology groups as abstract groups and considering the constraints on possible group homomorphisms. In other cases, use simplicial homology and your solution to part a.

   (i) The canonical quotient map $q : S^2 \to \mathbb{R}\mathrm{P}^2$.

   (ii) The inclusion of the equator $f : S^1 \to S^2$.

   (iii) The map $m : S^1 \to T$, where $T = S^1 \times S^1$, and m is the inclusion of the meridian $S^1 \times \{1\}$.

5. (a) Compute the singular homology groups the space S^2/A, where $A \subseteq S^2$ is a finite set of points.

   (b) **(Topology Qual, Sep 2016).** Let $Y = (S^1 \times S^1)/(S^1 \times \{1\})$ (i.e., collapse $S^1 \times \{1\}$ to a point) with the quotient topology. Find the homology of Y .

6. (a) Prove the following proposition.

   Proposition (Homology of a wedge sum). Let $\{X_\alpha\}$ be a collection of topological spaces, with basepoint $x_\alpha \in X_\alpha$ such that (X_α, x_α) is a good pair for each α. Let $\bigvee_\alpha X_\alpha$ be the wedge sum forrmed by identifying the basepoints x_α, and let $i_\alpha : X_\alpha \to \bigvee_\alpha X_\alpha$ be the inclusion map. Then for each n there is an isomorphism on homology

$$\oplus_\alpha (i_\alpha)_* : \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \xrightarrow{\cong} \widetilde{H}_n\left(\bigvee_\alpha X_\alpha\right)$$

   Hint: Consider the pair $(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\})$.

   (b) State the homology groups of the following spaces. No justification needed.

   (i) $\bigvee_k S^1$

   (ii) a once-punctured torus

   (iii) $S^1 \vee S^2 \vee S^3 \vee S^\infty$

   (iv) the wedge sum of $\mathbb{R}\mathrm{P}^2$ and a Mobius band