Final Exam

Math 592
26 April 2022
Jenny Wilson

Name: ______________________________

Instructions: This exam has 5 questions for a total of 30 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

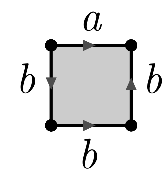
Jenny is available to answer questions.

Question	Points	Score
1	4	
2	6	
3	6	
4	5	
5	9	
Total:	30	

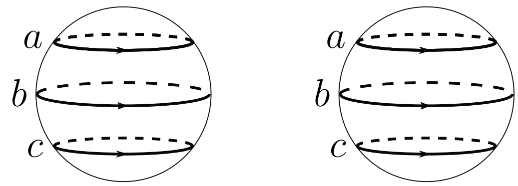
Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
  (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n\geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

1. (4 points) Let X be the space constructed from a square by the edge identifications shown. Let $A \subseteq X$ be the image of the loop labelled a. Prove that A is not a retract of X.



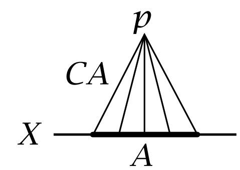
2. (6 points) Take two copies of the 2-sphere S^2, with three distinguished embedded loops a, b, c as shown. Let X be the quotient space obtained by identifying the two loops labelled a, the two loops labelled b, and the two loops labelled c. Calculate $\widetilde{H}_*(X)$.



Problem 2 ctd.

3. (6 points) Let A be a subspace of a topological space X. In this question, we will show that we can equate the relative homology groups $H_*(X, A)$ with the absolute (reduced) homology of a certain space even when (X, A) is not a good pair. This shows that relative homology is no more general than absolute homology.

Recall that the cone CA on A is the quotient $(A \times I)/(A \times \{0\})$. Let $X \cup CA$ denote the space obtained by identifying each point $a \in A \subseteq X$ with the image of $(a, 1)$ in CA, as shown. Let p denote the cone point, the image of $A \times \{0\}$ in CA.



To prove this result: justify each of the following isomorphisms of groups. Give complete statements of any theorems you cite and fully verify their hypotheses.

$$\widetilde{H}_k(X \cup CA) \overset{(i)}{\cong} H_k(X \cup CA, CA) \overset{(ii)}{\cong} H_k(X \cup CA \backslash \{p\}, CA \backslash \{p\}) \overset{(iii)}{\cong} H_k(X, A) \text{ for any } k \geq 0$$

Problem 3 ctd.

4. (5 points) Fix an integer $n \geq 1$. Let $\tilde{X}$ be a retract of $\mathbb{R}\mathrm{P}^{2n}$. Show that every regular (surjective) covering map $p : \tilde{X} \to X$ is a homeomorphism.

5. (9 points) For each of the following statements: if the statement is true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let X be a space, and let $A \subseteq B \subseteq X$ be subspaces such that B deformation retracts onto A. Then the quotient spaces X/A and X/B are homotopy equivalent.

   (b) Let X be a CW complex, and suppose that A is a subcomplex of X with the property that the inclusion map $A \hookrightarrow X$ is nullhomotopic. Then quotient map $X \to X/A$ is a homotopy equivalence.

   (c) Let X be a connected CW complex with no 2-cells. Then $\pi_1(X)$ is a (possibly trivial) free group.

   (d) Let X be a simply connected space. Then (up to homeomorphism) X admits a CW complex structure with no 1-cells.

(e) Let A be an abelian group, and suppose A has a generating set S with $|S| \leq 2g$. Then A is a the deck group of some regular cover of Σ_g.

(f) Suppose that a space X is a union $X = U \cup V \cup W$ of contractible open subsets U, V, W such that $U \cap V \cap W$ is nonempty and the intersections $U \cap V$, $V \cap W$, and $W \cap U$ are path-connected. Then X is simply connected.

(g) Let $p : \tilde{X} \to X$ be a covering map. Then $p_* : H_n(\tilde{X}) \to H_n(X)$ is injective for each n.

(h) Let S be a nonempty set. Recall that a topology on S is (in particular) a set of subsets of S. Define a category $\mathcal{C}$ where the objects are the topologies $\mathcal{T}$ on S, and there is one morphism $\mathcal{T} \to \mathcal{T}'$ if $\mathcal{T} \subseteq \mathcal{T}'$ and otherwise no morphisms $\mathcal{T} \to \mathcal{T}'$. Then $\mathcal{C}$ does not have a terminal object.

(i) Let $\mathcal{F}$ be the forgetful functor from the category $\underline{\text{Grp}}$ of groups and group homomorphisms to the category $\underline{\text{Set}}$ of sets and all functions. Then for all groups G, H, the map on morphisms $\mathcal{F} : \text{Hom}_{\underline{\text{Grp}}}(G, H) \to \text{Hom}_{\underline{\text{Set}}}(\mathcal{F}(G), \mathcal{F}(H))$ is injective. (Such a functor is called *faithful*).