Final Exam

Math 592
26 April 2022
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Name: ______________________________

Instructions: This exam has 5 questions for a total of 30 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

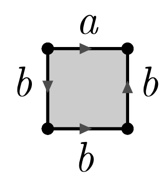
Jenny is available to answer questions.

Question	Points	Score
1	4	
2	6	
3	6	
4	5	
5	9	
Total:	30	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
  (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

1. (4 points) Let X be the space constructed from a square by the edge identifications shown. Let $A \subseteq X$ be the image of the loop labelled a. Prove that A is not a retract of X.



Solution. We could solve this problem by studying either $\pi_1(X)$ or $H_1(X)$. In this solution, we compute $\pi_1(X)$.

We verify by inspection that the edge identifications shown have the effect of identifying all four vertices of the square to a single point we call v. Hence, X has a CW complex structure where the 0-skeleton X^0 is a single point v, the 1-skeleton X^1 is the wedge of two loops a and b, and the 2-skeleton $X^2 = X$ is constructed by gluing single 2-disk along the word ab^{-3}.

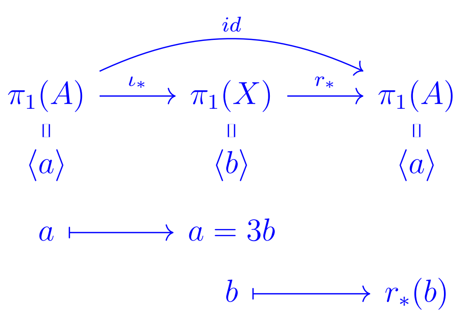
The fundamental group of X^1 is the free group on 2 generators. We proved that π_1 of a CW complex X is the quotient of $\pi_1(X^1)$ by a relation for each 2-cell. In this case,

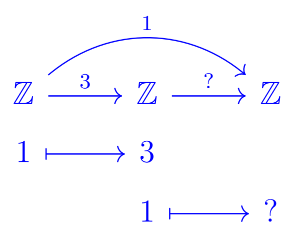
$$\pi_1(X) = \langle a, b \mid ab^{-3} \rangle = \langle a, b \mid a = b^3 \rangle \cong \mathbb{Z}.$$

Thus the fundamental group is a copy of $\mathbb{Z}$, generated by the loop b, and the element a is (written additively) $3b \in \langle b \rangle$.

We will use this calculation to show that A is not a retract of X. Suppose for the sake of contradiction that it is. By definition, this means that there exists a retraction, a map $r : X \to A$ satisfying $r \circ \iota = id_A$, where $\iota : A \hookrightarrow X$ is the inclusion of A.

By functoriality of π_1, this implies that $r_* \circ \iota_* = id_{\pi_1(A)}$. Now, A is a circle, with $\pi_1(A) \cong \mathbb{Z}$ generated by the loop a. The inclusion $\iota : A \to X$ induces on π_1 the inclusion of the subgroup generated by a, i.e., the inclusion of the subgroup $3\mathbb{Z} \subseteq \mathbb{Z} \cong \pi_1(X)$. We have a commuting diagram,



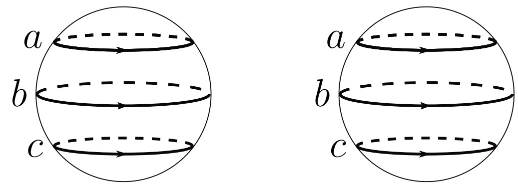


The crux of our contradiction is that the inclusion $3\mathbb{Z} \hookrightarrow \mathbb{Z}$ does not have a left inverse r_*. To see this, we consider the possibilities for the element $r_*(b)$. We know

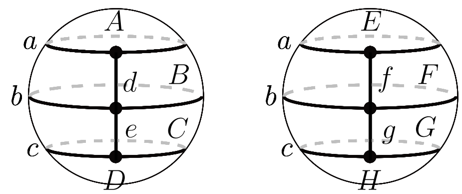
$$r_*(\iota_*(a)) = id(a) = a$$
$$r_*(3b) = a$$
$$3r_*(b) = a$$

But there is no element x in $\pi_1(A) = \langle a \rangle$ satisfying $3x = a$. We conclude that the retraction map r cannot exist, and A is not a retract of X.

2. (6 points) Take two copies of the 2-sphere S^2, with three distinguished embedded loops a, b, c as shown. Let X be the quotient space obtained by identifying the two loops labelled a, the two loops labelled b, and the two loops labelled c. Calculate $\widetilde{H}_*(X)$.



Solution. One solution would be to choose a CW structure on X and compute the cellular homology. For example, it has a CW structure with three 0-cells, seven 1-cells a, b, c, d, e, f, g, and eight 2-cells A, B, C, D, E, F, G, H as shown. The differentials are large but sparse matrices.



In this solution, however, we will use the Mayer–Vietoris long exact sequence. Let U be a neighbourhood of the three circles a, b, c that deformation retracts onto the three circles – this neighbourhood must exist because the circles form a CW subcomplex – and so $U \simeq S^1 \sqcup S^1 \sqcup S^1$. Let A be the union of U with the image of the first sphere in X, so $A \simeq S^2$. Let B be the union U with the image of the second sphere, so $B \simeq S^2$ and $A \cap B = U$. Then A and B are an open cover of X (in particular, their interiors cover X), so a Mayer–Vietoris sequence exists on the reduced homology groups (Homework #12 Problem 1):

$$\cdots \longrightarrow \widetilde{H}_{n+1}(A \cup B) \longrightarrow \widetilde{H}_n(A \cap B) \longrightarrow \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \longrightarrow \widetilde{H}_n(X) \longrightarrow \cdots$$

Since X is a 2-dimensional CW complex, we know $\widetilde{H}_k(X)$ can be nonzero only in degees $k = 0, 1, 2$. Moreover X is path-connected by construction, so $\widetilde{H}_0(X) = 0$.

To describe the long exact sequence in these degrees, we apply our calculations from class of the (reduced) homology of a sphere and the homology of a disjoint union.

$$\widetilde{H}_n(A) \cong \widetilde{H}_n(B) \cong \widetilde{H}_n(S^2) \cong \begin{cases} \mathbb{Z}, & n = 2 \\ 0, & \text{else} \end{cases}$$

$$H_n(A \cap B) = H_n(U) \cong H_n(S^1 \sqcup S^1 \sqcup S^1) \cong H_n(S^1) \oplus H_n(S^1) \oplus H_n(S^1) \cong \begin{cases} \mathbb{Z}^3, & n = 1 \\ \mathbb{Z}^3, & n = 0 \\ 0, & \text{else} \end{cases}$$

To compute reduced homology we reduce the rank in degree 0 by one (Homeowork #9 Problem 2),

$$\widetilde{H}_n(A \cap B) \cong \begin{cases} \mathbb{Z}^3, n = 1 \\ \mathbb{Z}^2, n = 0 \\ 0, \text{ else} \end{cases}$$

Then the we have following Mayer–Veitoris exact sequence of homology groups.

$$\begin{array}{ccccccccc}
 & & \cdots & \longrightarrow & \widetilde{H}_3(X) & \longrightarrow & \widetilde{H}_2(A \cap B) & & \\
 & & & & \| & & \| & & \\
 & & & & 0 & & 0 & & \\
\longrightarrow & \widetilde{H}_2(B) \oplus \widetilde{H}_2(A) & \longrightarrow & \widetilde{H}_2(X) & \longrightarrow & \widetilde{H}_1(A \cap B) & \longrightarrow & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) & \\
 & \| & & \| & & \| & & \| & \\
 & \mathbb{Z} \oplus \mathbb{Z} & & \widetilde{H}_2(X) & & \mathbb{Z}^3 & & 0 & \\
\longrightarrow & \widetilde{H}_1(X) & \longrightarrow & \widetilde{H}_0(A \cap B) & \longrightarrow & \widetilde{H}_0(B) \oplus \widetilde{H}_0(A) & \longrightarrow & \widetilde{H}_0(X) & \longrightarrow 0. \\
 & \| & & \| & & \| & & \| & \\
 & \widetilde{H}_1(X) & & \mathbb{Z}^2 & & 0 & & 0 &
\end{array}$$

This yields two short exact sequences

$$0 \longrightarrow \widetilde{H}_1(X) \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \widetilde{H}_2(X) \xrightarrow{q} \mathbb{Z}^3 \longrightarrow 0$$

The first sequence implies that $\widetilde{H}_1(X) \cong \mathbb{Z}^2$.

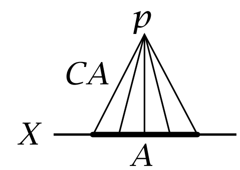
We claim that the second sequence implies $\widetilde{H}_2(X) \cong \mathbb{Z}^5$. In general, if the quotient of a short exact sequence of abelian groups is free abelian, then the sequence splits, which implies here that $\widetilde{H}_2(X) \cong \mathbb{Z}^3 \oplus \mathbb{Z}^2$. We will verify this result by hand in this case. Since X has a finite CW complex structure, we know $\widetilde{H}_2(X)$ is finitely generated. (We could also check this directly, since it is an extension of finitely generated abelian groups.) Thus by the structure theorem for finitely generated abelian groups, $\widetilde{H}_2(X) \cong \mathbb{Z}^n \oplus T$ for some integer $n \geq 0$ and some torsion subgroup T. Since the quotient $\mathbb{Z}^3$ is free abelian, the torsion subgroup T is in the kernel of q. But $\ker(q) \cong \mathbb{Z}^2$ is torsion-free, so $T = 0$. Thus $\widetilde{H}_2(X) \cong \mathbb{Z}^n$ for some n. Applying the rank-nullity theorem for free abelian groups (Homework #8 Problem 6) to the quotient map q we deduce $n = 5$.

We conclude

$$\widetilde{H}_n(X) \cong \begin{cases} \mathbb{Z}^5, \ n = 2 \\ \mathbb{Z}^2, \ n = 1 \\ 0, \ \text{else.} \end{cases}$$

3. (6 points) Let A be a subspace of a topological space X. In this question, we will show that we can equate the relative homology groups $H_*(X, A)$ with the absolute (reduced) homology of a certain space even when (X, A) is not a good pair. This shows that relative homology is no more general than absolute homology.

Recall that the cone CA on A is the quotient $(A \times I)/(A \times \{0\})$. Let $X \cup CA$ denote the space obtained by identifying each point $a \in A \subseteq X$ with the image of $(a, 1)$ in CA, as shown. Let p denote the cone point, the image of $A \times \{0\}$ in CA.



To prove this result: justify each of the following isomorphisms of groups. Give complete statements of any theorems you cite and fully verify their hypotheses.

$$\widetilde{H}_k(X \cup CA) \overset{(i)}{\cong} H_k(X \cup CA, CA) \overset{(ii)}{\cong} H_k(X \cup CA \backslash \{p\}, CA \backslash \{p\}) \overset{(iii)}{\cong} H_k(X, A) \text{ for any } k \geq 0$$

Solution. We will show that the isomorphism (i) follows from the long exact sequence of a pair. We proved:

> **Theorem (LES of a pair).** Let $Y \subseteq Z$ be spaces. Then there exists a long exact sequence on homology
>
> $$\cdots \longrightarrow \widetilde{H}_k(Y) \longrightarrow \widetilde{H}_k(Z) \longrightarrow H_k(Z, Y) \longrightarrow \widetilde{H}_{k-1}(Y) \longrightarrow \cdots$$

Applying this result to the pair $(X \cup CA, CA)$, we obtain a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_k(CA) \longrightarrow \widetilde{H}_k(X \cup CA) \longrightarrow H_k(X \cup CA, CA) \longrightarrow \widetilde{H}_{k-1}(CA) \longrightarrow \cdots$$

We claim that the space CA is contractible. The homotopy

$$\begin{aligned} F_s : A \times [0,1] &\longrightarrow A \times [0,1] \\ (a,t) &\longmapsto (a, (1-s)t) \end{aligned}$$

is a deformation retraction from $A \times [0,1]$ to $A \times \{0\}$. Since F_s preserves the subspace $A \times \{0\}$, it descends to a continuous homotopy on the quotient space CA (see Homework #2 Warm-up Problem 1) and defines a deformation retraction from CA to p. Thus CA is contractible.

We proved that the reduced homology of a contractible space vanishes in every degree. Thus our long exact sequence simplifies as follows for all k:

$$\cdots \longrightarrow 0 \longrightarrow \widetilde{H}_k(X \cup CA) \longrightarrow H_k(X \cup CA, CA) \longrightarrow 0 \longrightarrow \cdots$$

and we conclude that $\widetilde{H}_k(X \cup CA) \cong H_k(X \cup CA, CA)$ in every degree k.

We will now show that the map (ii) is an isomorphism using the excision theorem.

> **Theorem (excision).** Let $W \subseteq Y \subseteq Z$ be spaces such that the closure of W is contained in the interior of Y. Then the inclusion of pairs $(Z \setminus W, Y \setminus W) \hookrightarrow (Z, Y)$ induces an isomorphism on relative homology groups,
>
> $$H_k(Z \setminus W, Y \setminus W) \cong H_k(Z, Y) \qquad \text{for all } k.$$

To apply this result to the triple $\{p\} \subseteq CA \subseteq X \cup CA$, we must check that the closure of p is contained in the interior of CA. Let $q : A \times I \to CA$ be the quotient map. Since $q^{-1}(p) = A \times \{0\}$ is closed in $A \times I$, the set $\{p\}$ is closed by definition of the quotient topology, and $\{p\}$ is its own closure. The set $A \times [0, 0.5)$ is an open saturated subset, and so its image is an open subset of CA that contains p. We conclude that the closure of $\{p\}$ is contained in the interior of CA. By excision, then,

$$H_k(X \cup CA \setminus \{p\}, CA \setminus \{p\}) \cong H_k(X \cup CA, CA) \qquad \text{for all } k.$$

Finally we verify isomorphism (iii). By mild abuse of notation, write (a, t) for the image of the point $(a, t) \in A \times (0, 1]$ in $CA \setminus \{p\}$.

By Homework #10 Problem 3(b), if two maps of pairs $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs, then the induced maps $f_*, g_* : H_k(X, A) \to H_k(Y, B)$ are equal for all k. Suppose a map $f : (X, A) \to (Y, B)$ of pairs has a homotopy inverse $\overline{f} : (Y, B) \to (X, A)$ such that $f \circ \overline{f}$ and $\overline{f} \circ f$ are both homotopic to the identity through maps of pairs. Then (by functoriality) for any k the induced maps $f_* : H_k(X, A) \to H_k(Y, B)$ and $\overline{f}_* : H_k(Y, B) \to H_k(X, A)$ are two-sided inverses, in particular, f_* induces an isomorphism of relative homology groups.

So, to complete part (iii), it suffices to verify that the inclusion of pairs

$$\iota : (X, A) \hookrightarrow (X \cup CA \setminus \{p\}, CA \setminus \{p\})$$

is a homotopy equivalence. Consider the retraction $r : (X \cup CA \setminus \{p\}, CA \setminus \{p\}) \to (X, A)$ that fixes X and maps a point $(a, t) \in CA \setminus \{p\}$ to $a \in A = A \times \{1\}$. Then $r \circ \iota = id_{(X,A)}$. We will check that $\iota \circ r \simeq id_{(X \cup CA \setminus \{p\}, CA \setminus \{p\})}$ by constructing a deformation retraction of $(X \cup CA \setminus \{p\}, CA \setminus \{p\})$ onto (X, A) through maps of pairs. Let

$$\begin{aligned} F_s : (X \cup CA \setminus \{p\}, CA \setminus \{p\}) &\longrightarrow (X \cup CA \setminus \{p\}, CA \setminus \{p\}) \\ x &\longmapsto x & \text{for } x \in X \\ (a, t) &\longmapsto (a, (1 - s)t + s) & \text{for } (a, t) \in CA \setminus \{p\}. \end{aligned}$$

The map F_s is continuous since it is continuous on $X \sqcup \big(A \times (0, 1]\big)$ and constant on equivalence classes for all s. Moreover $F_0 = \mathrm{id}_{(X \cup CA \setminus \{p\}, CA \setminus \{p\})}$, $F_1 = \iota \circ r$, and $F_s(CA \setminus \{p\}) \subseteq CA \setminus \{p\}$ for all s. We conclude that ι is a homotopy equivalence as claimed, and

$$H_k(X \cup CA \setminus \{p\}, CA \setminus \{p\}) \cong H_k(X, A) \text{ for all } k.$$

4. (5 points) Fix an integer $n \geq 1$. Let $\tilde{X}$ be a retract of $\mathbb{R}\mathrm{P}^{2n}$. Show that every regular (surjective) covering map $p : \tilde{X} \to X$ is a homeomorphism.

Solution. We proved that a regular cover $p : \tilde{X} \to X$ is the quotient map to the orbit space of $\tilde{X}$ under the action of the deck group. Thus, it suffices to show that the deck group of any cover $p : \tilde{X} \to X$ is the trivial group.

We note that $\tilde{X}$ must be path-connected, since it is the image of the path-connected space $\mathbb{R}\mathrm{P}^{2n}$ under a (continuous, surjective) retraction map.

Consider a deck map $f : \tilde{X} \to \tilde{X}$. We proved (by uniqueness of lifts to covers) that every deck map of a connected cover is determined by the image of a single point. In particular, to show that f is the identity map, it suffices to show that f fixes a point.

Consider the map g defined to be the composition

$$\mathbb{R}\mathrm{P}^{2n} \xrightarrow{r} \tilde{X} \xrightarrow{f} \tilde{X} \xrightarrow{\iota} \mathbb{R}\mathrm{P}^{2n} \qquad \text{(where } r \text{ is the retraction and } \iota \text{ the inclusion).}$$

We showed that $\mathbb{R}\mathrm{P}^{2n}$ has a finite CW complex structure (Homework #2 Problem 1), and so we may apply the Lefschetz fixed point theorem to the map g. It states that, if the Lefschetz number $\tau(g) \neq 0$, the map g has a fixed point.

We calculated in class that $H_k(\mathbb{R}\mathrm{P}^{2n}) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}/2\mathbb{Z}, & 0 < k < 2n, k \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$

Since g is a map of path-connected spaces, it induces the identity map on $H_0(\mathbb{R}\mathrm{P}^{2n}) \cong \mathbb{Z}$, and so $\mathrm{Trace}\Big(g_* : H_0(\mathbb{R}\mathrm{P}^{2n}) \to H_0(\mathbb{R}\mathrm{P}^{2n})\Big) = 1$.

We defined the trace of a map of finitely generated abelian groups in terms of its restriction to the free part. For $k \neq 0$, the groups $H_k(\mathbb{R}\mathrm{P}^{2n})$ are torsion groups, and so the trace of the induced map g_* is zero. Thus, the Lefschetz number of g is

$$\begin{aligned} \tau(g) &= \sum_{k=0}^{\infty} (-1)^k \mathrm{Trace}\Big(g_* : H_k(\mathbb{R}\mathrm{P}^{2n}) \to H_k(\mathbb{R}\mathrm{P}^{2n})\Big) \\ &= (-1)^0(1) + 0 \\ &= 1. \end{aligned}$$

Thus g must fix a point. We stated in class in our proof of the Lefschetz fixed point theorem that f and $g = \iota \circ f \circ r$ have the same set of fixed points. Here we will check the relevant direction explicitly: that f fixes a point. Let x be a fixed point of g. Then $x \in im(g) \subseteq im(\iota) = \tilde{X}$. And

$$f(x) = [id_{\tilde{X}} \circ f \circ id_{\tilde{X}}](x) = [r \circ \iota \circ f \circ r \circ \iota](x) = [r \circ g \circ \iota](x) = r(g(x)) = r(x) = x.$$

We conclude that f fixes a point and the action of the deck group on $\tilde{X}$ is trivial.

Alternate solution. Argue that the Lefschetz fixed point theorem applies to $\tilde{X}$ (it is retract of finite simplicial complex since $\mathbb{R}\mathrm{P}^{2n}$ is) and that since $\tilde{X}$ is a retract of $\mathbb{R}\mathrm{P}^{2n}$, the homology group $H_k(\tilde{X})$ is a quotient of $H_k(\mathbb{R}\mathrm{P}^{2n})$ for all k.

5. (9 points) For each of the following statements: if the statement is true, write "True". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "False" without a counterexample.

   (a) Let X be a space, and let $A \subseteq B \subseteq X$ be subspaces such that B deformation retracts onto A. Then the quotient spaces X/A and X/B are homotopy equivalent.

   False. For example, let $X = S^1$. Let B be the complement of a point, and let $A \subseteq B$ be a point. Then B is homeomorphic to an open interval and deformation retracts onto A by a straight-line homotopy. But you proved on Homework #2 Problem 2(g) that S^1/B is contractible, whereas $S^1/A \cong S^1$ not contractible (its fundamental group is nonzero).

   (b) Let X be a CW complex, and suppose that A is a subcomplex of X with the property that the inclusion map $A \hookrightarrow X$ is nullhomotopic. Then quotient map $X \to X/A$ is a homotopy equivalence.

   False. It would be true if A were contractible, but that is a stronger assumption. Consider for example the inclusion $\partial D^2 \hookrightarrow D^2$ of the boundary of a 2-disk. This map is nullhomotopic (since D^2 is contractible), but you proved on Homework #1 Problem 4(c) that $D^2/\partial D^2 \cong S^2$. Unlike D^2, the sphere S^2 is not contractible; it has nonzero homology in degree 2.

   (c) Let X be a connected CW complex with no 2-cells. Then $\pi_1(X)$ is a (possibly trivial) free group.

   True. *Hint:* By Homework #4 Problem 2, $\pi_1(X)$ has a presentation with no relations.

   (d) Let X be a simply connected space. Then (up to homeomorphism) X admits a CW complex structure with no 1-cells.

   False. For example, consider the case that X is a tree. Then X is contractible (so $\pi_1(X) = 0$) but X is 1-dimensional, so every CW complex structure on X must have 1-cells. The invariance of dimension theorem (Homework #10 Problem 5(c)) implies that the dimension of a CW complex is a well-defined homeomorphism-invariant.

(e) Let A be an abelian group, and suppose A has a generating set S with $|S| \leq 2g$. Then A is a the deck group of some regular cover of Σ_g.

True. *Hint:* We proved that $\pi_1(\Sigma_g) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$. Since A is abelian and the one relation of $\pi_1(\Sigma_g)$ is a commutator, any map from the $2g$ generators of $\pi_1(\Sigma_g)$ to A extends to a group homomorphism. In particular, we can define a surjective homomorphism $\phi : \pi_1(\Sigma_g) \to A$ by choosing any surjection from the $2g$ generators onto S. We proved that the covering space of Σ_g correspsonding to its normal subgroup $\ker(\phi)$ is a regular cover with deck group $\pi_1(\Sigma_g)/\ker(\phi) \cong A$.

(f) Suppose that a space X is a union $X = U \cup V \cup W$ of contractible open subsets U, V, W such that $U \cap V \cap W$ is nonempty and the intersections $U \cap V$, $V \cap W$, and $W \cap U$ are path-connected. Then X is simply connected.

True. *Hint:* Apply van Kampen. For $x_0 \in U \cap V \cap W$, the group

$$\pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(W, x_0) = 0$$

surjects onto $\pi_1(X)$.

(g) Let $p : \tilde{X} \to X$ be a covering map. Then $p_* : H_n(\tilde{X}) \to H_n(X)$ is injective for each n.

False. The map induced on π_1 is injective, but this need not be true for the maps induced on homology. Consider, for example, the cover $p : S^2 \to \mathbb{RP}^2$. Then $H_2(S^2) \cong \mathbb{Z}$ but $H_2(\mathbb{RP}^2) = 0$, so the map p_* cannot inject.

(h) Let S be a nonempty set. Recall that a topology on S is (in particular) a set of subsets of S. Define a category $\mathcal{C}$ where the objects are the topologies $\mathcal{T}$ on S, and there is one morphism $\mathcal{T} \to \mathcal{T}'$ if $\mathcal{T} \subseteq \mathcal{T}'$ and otherwise no morphisms $\mathcal{T} \to \mathcal{T}'$. Then $\mathcal{C}$ does not have a terminal object.

False. The discrete topology $\mathcal{T}_d$ on S (i.e. $\mathcal{T}_d$ is the power set of S) is a terminal object. Given any topology $\mathcal{T}$ on S, there is containment $\mathcal{T} \subseteq \mathcal{T}_d$, and so there is a unique morphism $\mathcal{T} \to \mathcal{T}_d$.

(i) Let $\mathcal{F}$ be the forgetful functor from the category $\underline{\text{Grp}}$ of groups and group homomorphisms to the category $\underline{\text{Set}}$ of sets and all functions. Then for all groups G, H, the map on morphisms $\mathcal{F} : \text{Hom}_{\underline{\text{Grp}}}(G, H) \to \text{Hom}_{\underline{\text{Set}}}(\mathcal{F}(G), \mathcal{F}(H))$ is injective. (Such a functor is called *faithful*).

True. *Hint:* Two group homomorphisms $G \to H$ are distinct if and only if the underlying functions of sets are distinct. Even though $\mathcal{F}$ is not injective on objects, it is injective on hom-sets.