Midterm Exam I

Math 592
24 February 2022
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Name: ____________________

Instructions: This exam has 4 questions for a total of 25 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 90 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Question	Points	Score
1	4	
2	10	
3	2	
4	9	
Total:	25	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (unit n-sphere) (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{RP}^n$ real projective n-space
- $\mathbb{CP}^n$ complex projective n-space

1. (4 points) Let X be a topological space, and let $I = [0,1]$ be the unit interval with the usual topology. Recall that the cone CX on X is the space obtained by taking the product $X \times I$ and collapsing $X \times \{0\}$ to a point. Prove that the map

$$\begin{aligned} \underline{\text{Top}} &\longrightarrow \underline{\text{Top}} \\ X &\longmapsto CX \end{aligned}$$

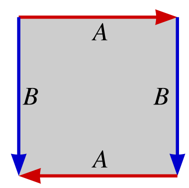
defines a functor. Here Top is the category of topological spaces and continuous maps.

2. For each of the following spaces X, give a presentation for the fundamental group.

   You do not need to give rigorous proofs, but please show your work in enough detail that I can understand and check your steps.

   (a) (2 points) $X = (\mathbb{R}P^2 \times \mathbb{R}P^3) \vee \mathbb{R}P^4$

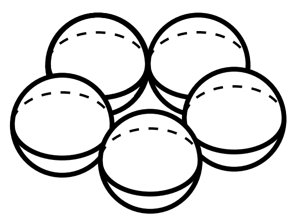
   (b) (2 points) The space X is obtained from a Klein bottle (pictured) by gluing a second 2-disk along $ABAB^{-1}$.



   (c) (2 points) Let Y be a CW complex structure on a 10-disk, and X its 6-skeleton.

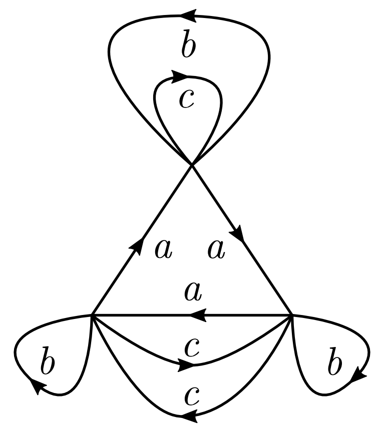
(d) (2 points) Let $S \subseteq \mathbb{R}^2$ be a 5-point subset. Let X be the quotient $\mathbb{R}^2/S$, glueing together the five points.

(e) (2 points) Let X be the "necklace" of five 2-spheres as shown. Each sphere is glued to each neighbour at a point.



3. (2 points) Let $S^1 \vee S^1 \vee S^1$ be the wedge of circles a, b, c, so we may identify its fundamental group with the free group $F_{\{a,b,c\}}$ on the set $\{a, b, c\}$. Consider the following connected cover of $S^1 \vee S^1 \vee S^1$. Find free generators for the image of its fundamental group in $F_{\{a,b,c\}}$.

   You do not need to give a rigorous proof, but please briefly explain your steps.



4. (9 points) For each of the following statements: if the statement is true, write "True". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "False" without a counterexample.

   (a) Let X, Y be spaces, and $A \subseteq X$ a subspace. Suppose $f : X \to Y$ is a homotopy equivalence. Then $f|_A : A \to f(A)$ is a homotopy equivalence.

   (b) Let F be a covariant functor from the category of topological spaces and continuous maps, to the category of abelian groups and group homomorphisms. If f is a homeomorphism, then $F(f)$ is an isomorphism of abelian groups.

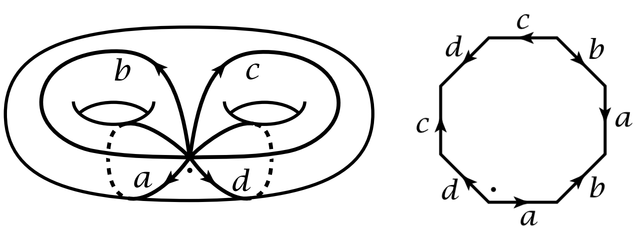
   (c) Let X be a CW complex, and suppose $f : X \to Y$ is a continuous map whose restriction $f|_{X^1}$ to the 1-skeleton is nullhomotopic. Then f_* induces the trivial map on fundamental group.

   (d) If a continuous map of path-connected spaces $f : X \to Y$ is surjective, then the induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is surjective.

(e) Suppose a certain space X decomposes as a union of three open **contractible** subsets $X = A \cup B \cup C$ with $A \cap B \cap C \neq \varnothing$. Then $\pi_1(X) = 0$.

(f) There does not exist a connected CW complex Y with $\pi_1(Y) \cong \langle a, b, c \mid abca, bcbc \rangle$.

(g) Recall that $\pi_1(\Sigma_2, x_0)$ is generated by the four loops a, b, c, d shown. Any map of sets from the set $\{a, b, c, d\}$ to any abelian group A extends uniquely to a group homomorphism $\pi_1(\Sigma_2, x_0) \to A$.



(h) Every continuous map from S^1 to $S^1 \vee S^1$ is nullhomotopic.

(i) Let $n \geq 2$. Every continuous map from S^n to $S^1 \vee S^1$ is nullhomotopic.