Midterm Exam II

Math 592
24 March 2022
Jenny Wilson

Name: ______________________________

Instructions: This exam has 4 questions for a total of 20 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 60 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	6	
2	4	
3	5	
4	5	
Total:	20	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
  (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

1. Consider the wedge $S^1 \vee S^1$ of two circles labelled a and b, respectively, based at its wedge point x_0. Identify its fundamental group with the free group $F_{\{a,b\}}$ on the set $\{a, b\}$. Consider the connected based covers of $S^1 \vee S^1$ associated to the following subgroups of $F_{\{a,b\}}$. For each cover,

   (i) Draw and label the cover, and label its basepoint.

   (ii) State the degree (i.e., the number of sheets).

   (iii) State whether it is regular.

   (iv) State the deck group, as an abstract group.

   (v) Describe (in words or your favourite notation for permutations) how $a \in F_{\{a,b\}}$ acts on the fibre $p^{-1}(x_0)$.

   No justification required.

   (a) (3 points) The subgroup $\langle b^2 \rangle$.

   (b) (3 points) The kernel of the homomorphism

$$\begin{aligned} \varphi : F_{\{a,b\}} &\longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ a &\longmapsto (1, 0) \\ b &\longmapsto (1, 1) \end{aligned}$$

2. (4 points) Define Smith normal form, and briefly describe the steps of how to use the existence of the Smith normal form of an integer matrix to calculate homology. Illustrate your explanation by calculating the homology of the middle group in the following chain complex.

$$\longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 2 & 4 \\ -2 & -4 \\ 2 & 4 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -3 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow$$

3. (5 points) Fix $d \geq 1$. Let X denote a d-dimensional Δ-complex, and suppose that X is homotopy equivalent to a d-sphere. Let Y denote the $(d-1)$-skeleton of X. Prove that

$$\widetilde{H}_i(Y) = 0 \qquad \text{for } i \neq d-1$$

and $\widetilde{H}_{d-1}(Y)$ is generated by cycles equal to the boundaries of d-simplices of X,

$$\{\partial\Delta_i \mid \Delta_i \text{ a } d\text{-simplex of } X\} \subseteq C_{d-1}(Y).$$

Problem 3 continued.

4. (5 points) For each of the following statements: if the statement is true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let F_n denote the free group of rank n. There does not exist a finite-index subgroup of F_3 isomorphic to F_5.

   (b) Let F_n denote the free group of rank n. There does not exist a finite-index subgroup of F_4 isomorphic to F_5.

   (c) Let $n > 0$. Any Δ-complex X that is homotopy equivalent to $S^n \vee S^n \vee S^n$ must have at least three n-simplices.

   (d) There does not exist a 2-dimensional Δ-complex X such that $H_1(X) \cong \mathbb{Z}/4\mathbb{Z}$.

   (e) There does not exist a 2-dimensional Δ-complex X such that $H_2(X) \cong \mathbb{Z}/4\mathbb{Z}$.