Midterm Exam II

Math 592
24 March 2022
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Name: ____

Instructions: This exam has 4 questions for a total of 20 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 90 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	6	
2	4	
3	5	
4	5	
Total:	20	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
  (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

1. Consider the wedge $S^1 \vee S^1$ of two circles labelled a and b, respectively, based at its wedge point x_0. Identify its fundamental group with the free group $F_{\{a,b\}}$ on the set $\{a, b\}$. Consider the connected based covers of $S^1 \vee S^1$ associated to the following subgroups of $F_{\{a,b\}}$. For each cover,

   (i) Draw and label the cover, and label its basepoint.

   (ii) State the degree (i.e., the number of sheets).

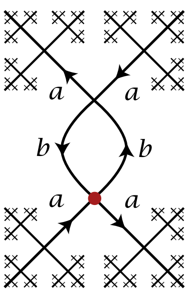
   (iii) State whether it is regular.

   (iv) State the deck group, as an abstract group.

   (v) Describe (in words or your favourite notation for permutations) how $a \in F_{\{a,b\}}$ acts on the fibre $p^{-1}(x_0)$.

   No justification required.

   (a) (3 points) The subgroup $\langle b^2 \rangle$.



   (i) We seek a cover $\tilde{X}$ with fundamental group $\mathbb{Z}$ generated by a loop labelled b^2. This cover is shown, with one of the two valid choices of basepoints labelled by a dot.

   (ii) The degree of the cover is the number of vertices of $\tilde{X}$. It is **(countably) infinite**.

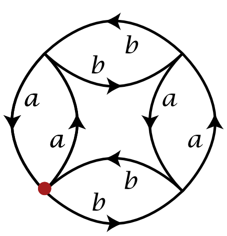
   (iii) The graph is **not regular**, by inspection, the deck group is not transitive on vertices. We will also see below its deck group has cardinality less than the degree of the cover.

   (iv) **The deck group is** $\mathbb{Z}/2\mathbb{Z}$. By inspection, the only nontrivial deck map is $180°$ rotation of the graph in the plane of the page. Alternatively, we could calculate the normalizer $N(\langle b^2 \rangle) = \langle b \rangle$, and $\langle b \rangle / \langle b^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

   (v) The element a acts on a vertex x by mapping x one edge length along an edge labelled a, against the direction of the arrow. E.g. in the top left corner of the graph, each vertex moves one step south-east.

   (b) (3 points) The kernel of the homomorphism

$$\begin{aligned} \varphi : F_{\{a,b\}} &\longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ a &\longmapsto (1, 0) \\ b &\longmapsto (1, 1) \end{aligned}$$



   (i) We can construct $\tilde{X}$ using the method you developed in Homework 6 Problem 6(a). Any choice of vertex for the basepoint is equivalent.

   (ii) The degree is $|p^{-1}(x_0)| = \mathbf{4}$.

   (iii) The graph is **regular**; the kernel of a homomorphism is always a normal subgroup.

   (iv) **The deck group is** $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since φ surjects, it follows $F_{\{a,b\}}/\ker(\varphi) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   (v) The element a acts by an involution, interchanging each vertex with the vertex directly above or below it.

2. (4 points) Define Smith normal form, and briefly describe the steps of how to use the existence of the Smith normal form of an integer matrix to calculate homology. Illustrate your explanation by calculating the homology of the middle group in the following chain complex.

$$\longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 2 & 4 \\ -2 & -4 \\ 2 & 4 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -3 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow$$

Solution. Call the leftmost matrix A and the rightmost B. The first step is to put A and B in Smith normal form. This means we use row and column operations (invertible over $\mathbb{Z}$) to transform the matrices into the form

$$\begin{bmatrix} \alpha_1 & 0 & 0 & & \cdots & & 0 \\ 0 & \alpha_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & \alpha_r & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{bmatrix}$$

where the diagonal entries α_i satisfy $\alpha_i | \alpha_{i+1}$ for all $1 \le i \le r$. The nonzero entries are the *invariant factors* and the number of nonzero entries is the *rank* of the matrix. We carry out these transformations for A and B:

$$A = \begin{bmatrix} 2 & 4 \\ -2 & -4 \\ 2 & 4 \end{bmatrix} \overset{\substack{R_2 \mapsto R_2 + R_1 \\ R_3 \mapsto R_3 - R_1}}{\rightsquigarrow} \begin{bmatrix} 2 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \overset{C_2 \mapsto C_2 - 2C_1}{\rightsquigarrow} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -3 \end{bmatrix} \overset{\substack{C_1 \mapsto C_1 - 2C_2 \\ C_3 \mapsto C_3 + C_2}}{\rightsquigarrow} \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & -3 \end{bmatrix} \overset{R_3 \mapsto R_3 + R_1}{\rightsquigarrow} \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \overset{\textit{col swap}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

Let d be the rank of the codomain of A (which is the domain of B). Then by Homework #9 Problem 1, the homology is given by the formula

$$\begin{aligned} \ker(B)/\mathrm{im}(A) &\cong \mathbb{Z}^{d - \mathrm{rank}(A) - \mathrm{rank}(B)} \oplus \bigoplus_{\text{invariant factors } \alpha \text{ of } A} \mathbb{Z}/\alpha\mathbb{Z} \\ &\cong \mathbb{Z}^{3-1-2} \oplus \mathbb{Z}/2\mathbb{Z} \\ &\cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

3. (5 points) Fix $d \geq 1$. Let X denote a d-dimensional Δ-complex, and suppose that X is homotopy equivalent to a d-sphere. Let Y denote the $(d-1)$-skeleton of X. Prove that

$$\widetilde{H}_i(Y) = 0 \qquad \text{for } i \neq d-1$$

and $\widetilde{H}_{d-1}(Y)$ is generated by cycles equal to the boundaries of d-simplices of X,

$$\{\partial \Delta_i \mid \Delta_i \text{ a } d\text{-simplex of } X\} \subseteq C_{d-1}(Y).$$

This problem appeared on the Aug 2021 QR Exam. See the official solutions for an alternate argument using the long exact sequence of a pair.

Solution. View the augmented simplicial chain complex $C_*(Y)$ as a sub-chain complex of the augmented simplicial chain complex $C_*(X)$.

$$\begin{tikzcd}
{} \arrow[r] & C_d(X) \arrow[r, "\partial_d^X"] & C_{d-1}(X) \arrow[r, "\partial_{d-1}^X"] \arrow[d, equal] & C_{d-2}(X) \arrow[r, "\partial_{d-2}^X"] \arrow[d, equal] & \cdots \arrow[r, "\partial_2^X"] & C_1(X) \arrow[r, "\partial_1^X"] \arrow[d, equal] & C_0(X) \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & 0 \arrow[r, "\partial_d^Y"] & C_{d-1}(Y) \arrow[r, "\partial_{d-1}^Y"] & C_{d-2}(Y) \arrow[r, "\partial_{d-2}^Y"] & \cdots \arrow[r, "\partial_2^Y"] & C_1(Y) \arrow[r, "\partial_1^Y"] & C_0(Y) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

For $i \leq d-2$, we have equality of maps (including equality of domains and codomains) $\partial_{i+1}^X = \partial_{i+1}^Y$ and $\partial_i^X = \partial_i^Y$,

$$\begin{tikzcd}
C_{i+1}(X) \arrow[r, "\partial_{i+1}^X"] \arrow[d, equal] & C_i(X) \arrow[r, "\partial_i^X"] \arrow[d, equal] & C_{i-1}(X) \arrow[d, equal] \\
C_{i+1}(Y) \arrow[r, "\partial_{i+1}^Y"] & C_i(Y) \arrow[r, "\partial_i^Y"] & C_{i-1}(Y)
\end{tikzcd}$$

so $\widetilde{H}_i(Y) = \widetilde{H}_i(X)$ for $i \leq d-2$. Since X is homotopy equivalent to a d-sphere, these groups vanish.

When $i \geq d > \dim(Y)$, the simplicial chain groups $C_i(Y)$ vanish and $\widetilde{H}_i(Y) = 0$.

Let $i = d-1$. Since X is homotopy equivalent to a d-sphere, $\widetilde{H}_{d-1}(X) = 0$ and the simplicial chain complex $C_*(X)$ is exact at $C_{d-1}(X)$. This means that the kernel of ∂_{d-1}^X is equal to the image of ∂_d^X. By definition of the boundary map this image of ∂_d^X is generated by $\{\partial \Delta_i \mid \Delta_i \text{ a } d\text{-simplex of } X\}$.

Since $C_d(Y) = 0$, the homology of the chain complex $C_*(Y)$ at $C_{d-1}(Y)$ is equal to

$$\ker(\partial_{d-1}^Y) = \ker(\partial_{d-1}^X),$$

and we conclude that $\widetilde{H}_{d-1}(Y)$ is the submodule of $C_{d-1}(Y) = C_{d-1}(X)$ spanned by $\{\partial \Delta_i \mid \Delta_i \text{ a } d\text{-simplex of } X\}$.

Problem 3 continued.

4. (5 points) For each of the following statements: if the statement is true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let F_n denote the free group of rank n. There does not exist a finite-index subgroup of F_3 isomorphic to F_5.

   False. By the Schreier index formula (Homework #7 Problem 1(d)), any index-2 subgroup of F_3 will be a free group of rank 5. To construct such a subgroup, we can (i) draw a 2-sheeted cover of $S^1 \vee S^1 \vee S^1$ and write down a generating set for the image of its fundamental group, or (ii) find a surjective homomorphism from F_3 to a group of order 2, e.g., consider the homomorphism $F_3 \to \mathbb{Z}/2\mathbb{Z}$ taking all 3 free generators of F_3 to the generator; its kernel is a free group of rank 5.

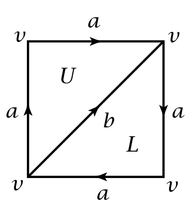
   (b) Let F_n denote the free group of rank n. There does not exist a finite-index subgroup of F_4 isomorphic to F_5.

   True. *Hint*: By the Schreier index formula (Homework #7 Problem 1(d)), any finite-index subgroup of F_4 has rank congruent to 1 mod 3.

   (c) Let $n > 0$. Any Δ-complex X that is homotopy equivalent to $S^n \vee S^n \vee S^n$ must have at least three n-simplices.

   True. *Hint*: Since homology is a homotopy invariant, $H_n(X) \cong H_n(S^n \vee S^n \vee S^n) \cong \mathbb{Z}^3$. But $H_n(X)$ is a subquotient of the group $C_n(X)$, so $C_n(X)$ must have rank at least 3.

   (d) There does not exist a 2-dimensional Δ-complex X such that $H_1(X) \cong \mathbb{Z}/4\mathbb{Z}$.



   False. For example, the following Δ-complex X was constructed so that $\pi_1(X) \cong \mathbb{Z}/4\mathbb{Z}$, by gluing the 2-disk $U \cup L$ along the word a^4. Hence $H_1(X) \cong \pi_1(X)^{ab} \cong \mathbb{Z}/4\mathbb{Z}$. We could also verify its first homology by direct calculation of simplicial homology.

   (e) There does not exist a 2-dimensional Δ-complex X such that $H_2(X) \cong \mathbb{Z}/4\mathbb{Z}$.

   True. *Hint*: Since X has no 3-simplices, $C_3(X) = 0$. Then

$$H_2(X) = \frac{\ker(\partial_2)}{\mathrm{im}(\partial_3)} = \frac{\ker(\partial_2)}{0} \cong \ker(d_2).$$

   But $\ker(d_2)$ is a subgroup of the free abelian group $C_2(X)$ and is thus free abelian.