Midterm Exam I

Math 592
24 February 2022
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Name: ____________________

Instructions: This exam has 4 questions for a total of 25 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 90 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Question	Points	Score
1	4	
2	10	
3	2	
4	9	
Total:	25	

Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
  (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n\geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ complex projective n-space

1. (4 points) Let X be a topological space, and let $I = [0,1]$ be the unit interval with the usual topology. Recall that the cone CX on X is the space obtained by taking the product $X \times I$ and collapsing $X \times \{0\}$ to a point. Prove that the map

$$\begin{aligned} \underline{\text{Top}} &\longrightarrow \underline{\text{Top}} \\ X &\longmapsto CX \end{aligned}$$

defines a functor. Here $\underline{\text{Top}}$ is the category of topological spaces and continuous maps.

Solution. Call the functor C. First we must define C on morphisms. Let $f : X \to Y$ be a continuous map of spaces. Then we obtain a map

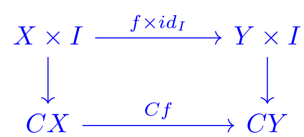
$$\begin{aligned} f \times id_I : X \times I &\longrightarrow Y \times I \\ (x,t) &\longmapsto (f(x), t). \end{aligned}$$

The map $f \times id_I$ is continuous on the product topology because it is a product of continuous maps. Consider the composition of $f \times id_I$ with the quotient map

$$X \times I \xrightarrow{f \times id_I} Y \times I \longrightarrow \frac{Y \times I}{Y \times \{0\}} = CY$$

It restricts to a map $X \times \{0\} \longrightarrow Y \times \{0\} \longrightarrow *$

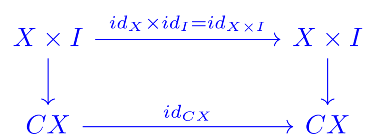
Thus the subspace $X \times \{0\}$ maps to a point in CY. By the universal property of the quotient topology, the map $X \times I \to CY$ factors uniquely through a continuous map from the quotient $(X \times I)/(X \times \{0\}) = CX$. Let $Cf : CX \to CY$ be this map.



Note that the universal property of the quotient implies Cf is the *unique* map completing this commuting square.

We will show the assignment $f \mapsto Cf$ defines a covariant functor. We must check two axioms: the identity axiom, and the composition axiom.

First we observe that, if $f = id_X : X \to X$, then we have a commuting square as shown. Thus $C(id_X) = id_{CX}$ as needed.

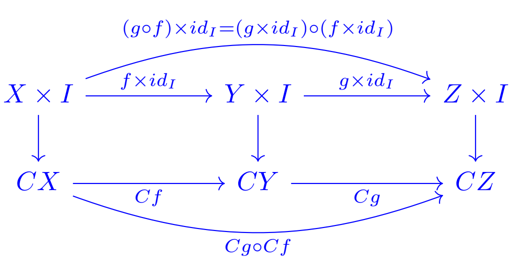


Now consider continuous maps $f : X \to Y$ and $g : Y \to Z$. Then

$$[(g \times id_I) \circ (f \times id_I)](x,t) = (g \times id_I)(f(x), t) = (g \circ f(x), t) = [(g \circ f) \times id_I](x,t)$$

We obtain the commuting diagram shown. Since the map $Cg \circ Cf$ makes the outer rectangle commute, we conclude that

$$C(g \circ f) = Cg \circ Cf.$$



This concludes the proof that C is a functor.

Alternative: We could also observe that the formula for Cf on equivalence classes is $Cf([(x,t)]) = [(f(x),t)]$, and then check the two axioms hold pointwise for $[(x,t)] \in CX$.

2. For each of the following spaces X, give a presentation for the fundamental group.

   You do not need to give rigorous proofs, but please show your work in enough detail that I can understand and check your steps.

   (a) (2 points) $X = (\mathbb{RP}^2 \times \mathbb{RP}^3) \vee \mathbb{RP}^4$

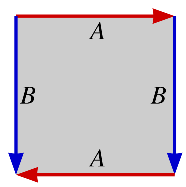
   Solution.

$$\begin{aligned}\pi_1(X) &\cong \pi_1\Big((\mathbb{RP}^2 \times \mathbb{RP}^3) \vee \mathbb{RP}^4\Big)\\ &\cong \Big(\pi_1(\mathbb{RP}^2) \times \pi_1(\mathbb{RP}^3)\Big) * \pi_1(\mathbb{RP}^4)\\ &\cong \Big(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\Big) * \mathbb{Z}/2\mathbb{Z}\end{aligned}$$

   and so

$$\pi_1(X) = \langle a, b, c \mid ab = ba, a^2 = b^2 = c^2 = 1\rangle.$$

   (b) (2 points) The space X is obtained from a Klein bottle (pictured) by gluing a second 2-disk along $ABAB^{-1}$.



   Solution. Hatcher Proposition 1.26 (Homework 4) implies that, because the Klein bottle has a disk glued along the word $ABAB^{-1}$, this loop is trivial in π_1. Proposition 1.26 further implies that gluing an additional disk along this loop will not change π_1. Hence X has the same fundamental group as the Klein bottlle,

$$\pi_1(X) \cong \langle A, B \mid ABAB^{-1}\rangle.$$

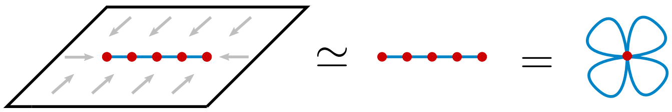
   (c) (2 points) Let Y be a CW complex structure on a 10-disk, and X its 6-skeleton.

   Solution. Let W denote the (common) 2-skeleton of X and Y. You proved (Homework 4, Problem 2) that the inclusions $W \hookrightarrow X$ and $W \hookrightarrow Y$ induce isomorphisms on π_1. Thus $\pi_1(X) \cong \pi_1(W) \cong \pi_1(Y)$. However, Y is contractible, so its fundamental group is trivial.

$$\pi_1(X) = 0 \cong \langle \mid \rangle.$$

(d) (2 points) Let $S \subseteq \mathbb{R}^2$ be a 5-point subset. Let X be the quotient $\mathbb{R}^2/S$, glueing together the five points.

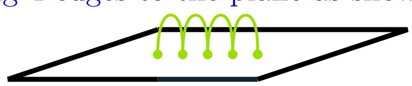
Solution. Let X be the quotient of the plane by the 5 points labelled in red. The plane deformation retracts onto the line segment marked in blue, and (since this homotopy respects the equivalence relation) it descends to a well-defined deformation retraction on the quotient space X.



Thus, X is homotopy equivalent to a wedge of 4 circles. We conclude

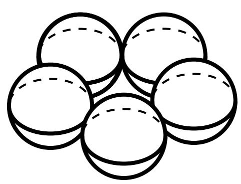
$$\pi_1(X) \cong \langle a, b, c, d \mid \rangle.$$

Alternate Solution. You proved (Homework 2) that the quotient of a CW complex by a contractible subcomplex is a homotopy equivalence. Consider the space Y obtained by gluing 4 edges to the plane as shown.

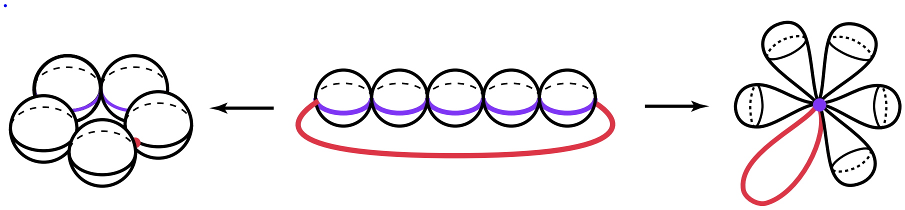


Since these edges form a contractible subcomplex, the quotient map collapsing them to a point is a homotopy equivalence. Thus, Y is homotopy equivalent to X. On the other hand, collapsing the plane to a point is a homotopy equivalence from Y to a wedge of 4 circles.

(e) (2 points) Let X be the "necklace" of five 2-spheres as shown. Each sphere is glued to each neighbour at a point.



Solution. You proved (Homework 2) that the quotient of a CW complex by a contractible subcomplex is a homotopy equivalence. We will apply this principle twice, to show that X is homotopy equivalent to a wedge of a circle and five 2-spheres.

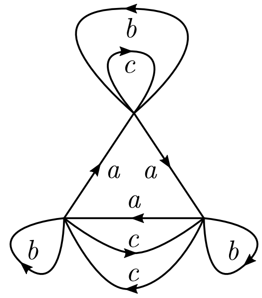


Thus $\pi_1(X) \cong \mathbb{Z} * 0 * 0 * 0 * 0 * 0 \cong \mathbb{Z}$, and $\pi_1(X)$ has presentation

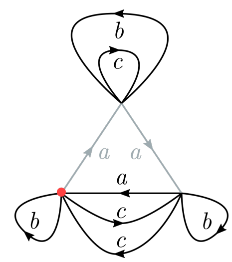
$$\pi_1(X) \cong \langle a \mid \rangle.$$

3. (2 points) Let $S^1 \vee S^1 \vee S^1$ be the wedge of circles a, b, c, so we may identify its fundamental group with the free group $F_{\{a,b,c\}}$ on the set $\{a, b, c\}$. Consider the following cover $\tilde{X}$ of $S^1 \vee S^1 \vee S^1$. Find free generators for the image of its fundamental group in $F_{\{a,b,c\}}$.

   You do not need to give a rigorous proof, but please briefly explain your steps.



Solution. Our first step is to choose a basepoint x_0 of $\tilde{X}$ (marked by a red dot) and find a free generating set of $\pi_1(\tilde{X}, x_0)$. To do this, we choose a maximal tree, that is, a contractible subgraph containing every vertex of $\tilde{X}$. A choice of maximal tree T is shown in gray.

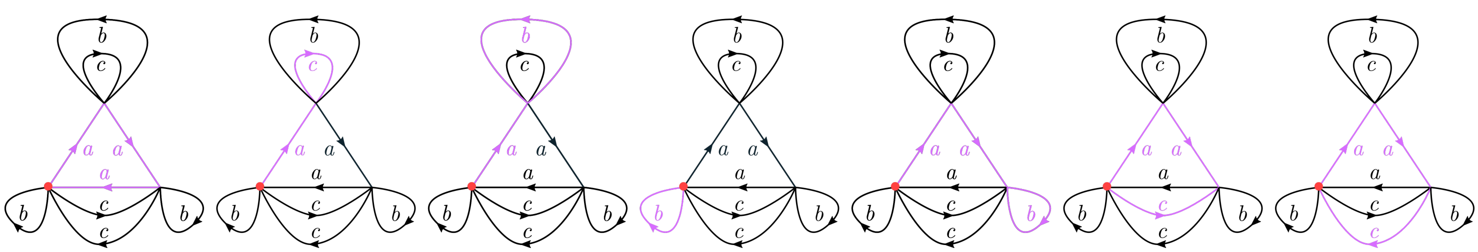


The quotient $\tilde{X}/T$ is wedge of 7 circles, one corresponding to each edge of $\tilde{X}$ not contained in T. Since T is a contractible subcomplex, this quotient is a homotopy equivalence, and we see that $\pi_1(\tilde{X}, x_0)$ is the free group F_7. To find a free generating set, we lift the 7 cirlces to $\tilde{X}$. This means, for each edge $e \in \tilde{X}$ not in T, we must find a loop that travels from x_0 through T to e, traverses e, and then returns through T to x_0. We may make either choice of orientation on the loop.

Seven loops are shown below. By reading off the words given by edge labelling, we find that the image of $\pi_1(\tilde{X}, x_0)$ in $F_{\{a,b,c\}}$ is an isomorphic copy of F_7 freely generated by the words

$$a^3, aca^{-1}, aba^{-1}, b, a^2ba^{-2}, a^2c^{-1}, a^2c.$$

A different choice of basepoint and maximal tree may result in a different generating set.



4. (9 points) For each of the following statements: if the statement is true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let X, Y be spaces, and $A \subseteq X$ a subspace. Suppose $f : X \to Y$ is a homotopy equivalence. Then $f|_A : A \to f(A)$ is a homotopy equivalence.

   False. The analogous statement is true for homeomorphisms, but not homotopy equivalences. Consider, for example, the constant map $f : \mathbb{R}^2 \to \{*\}$. Then f is a homotopy equivalence, but if we restrict it to some non-contractible subspace of $\mathbb{R}^2$ such as the unit circle $A = S^1$, the restriction $f|_A$ is not a homotopy equivalence.

   (b) Let F be a covariant functor from the category of topological spaces and continuous maps, to the category of abelian groups and group homomorphisms. If f is a homeomorphism, then $F(f)$ is an isomorphism of abelian groups.

   True. *Hint:* In both categories an isomorphism $f : X \to Y$ is precisely a morphism with an inverse morphism $f^{-1} : Y \to X$ in $\mathscr{C}$ satisfying $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$. Then by definition of functoriality, all functors must map isomorphisms to isomorphisms.

   (c) Let X be a CW complex, and suppose $f : X \to Y$ is a continuous map whose restriction $f|_{X^1}$ to the 1-skeleton is nullhomotopic. Then f_* induces the trivial map on fundamental group.

   True. *Hint:* By assumption the composite $X^1 \hookrightarrow X \to Y$ induces the zero map on π_1. But you proved that the inclusion $X^1 \hookrightarrow X$ induces a surjection on π_1.

   (d) If a continuous map of path-connected spaces $f : X \to Y$ is surjective, then the induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is surjective.

   False. For example, consider the quotient map of the interval

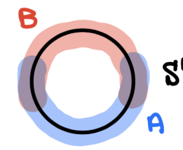
   $$f : I \to I/\{0,1\} \cong S^1$$

   gluing its endpoints together to form a circle. This map is surjective, but the induced map on π_1 is the inclusion of 0 into $\mathbb{Z}$.

   Alternate example: Your favourite multi-sheeted covering space map.

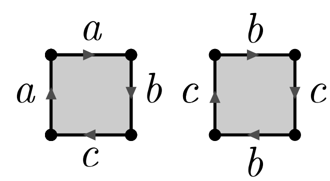
(e) Suppose a certain space X decomposes as a union of three open **contractible** subsets $X = A \cup B \cup C$ with $A \cap B \cap C \neq \varnothing$. Then $\pi_1(X) = 0$.

False. By van Kampen, this would be true if we knew the pairwise intersections $A \cap B$, $B \cap C$, and $A \cap C$ were path-connected. Without this assumption, however, the statement may not hold. For example, consider the decomposition of the circle S^1 into the sets A and B shown, and let C be one component of $A \cap B$.

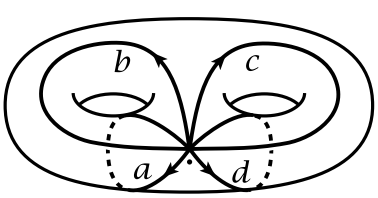


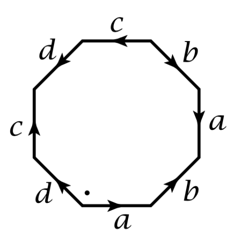
(f) There does not exist a connected CW complex Y with $\pi_1(Y) \cong \langle a, b, c \mid abca, bcbc \rangle$.

False. We can construct such a CW complex Y using 1 vertex, 3 edges a, b, c, and a 2-disk for each relator, as shown.



(g) Recall that $\pi_1(\Sigma_2, x_0)$ is generated by the four loops a, b, c, d shown. Any map of sets from the set $\{a, b, c, d\}$ to any abelian group A extends uniquely to a group homomorphism $\pi_1(\Sigma_2, x_0) \to A$.





True. *Hint:* you showed $\pi_1(\Sigma_2, x_0)$ is $\langle\, a, b, c, d \mid [a, b][c, d] \,\rangle$. Since the relator is a product of commutators, it will be satisfied by any four elements of any abelian group. Thus any map $\{a, b, c, d\} \to A$ extends to a group homomorphism.

(h) Every continuous map from S^1 to $S^1 \vee S^1$ is nullhomotopic.

False. For example, consider the inclusion of one of the two circles into the wedge sum $S^1 \to S^1 \vee S^1$. The induced map on π_1 is nonzero (it is the inclusion of one of the free factors of $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$) and thus cannot be nullhomotopic.

(i) Let $n \geq 2$. Every continuous map from S^n to $S^1 \vee S^1$ is nullhomotopic.

True. *Hint:* Let $f : S^n \to S^1 \vee S^1$. Since $\pi_1(S^n) = 0$, our lifting criterion implies that f factors through the universal cover $\tilde{X}$ of $S^1 \vee S^1$. But we have seen that the universal cover is a tree, hence contractible.