Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (n-sphere) (we sometimes view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n\geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ complex projective n-space

Practice problems

1. Describe all (based) isomorphism classes of regular 3-sheeted covers of $S^1 \vee S^1$.
2. The *Euler number* of a finite graph X is the number of vertices of X minus the number of edges of X.
   (a) Suppose X is a finite connected graph with Euler number $\chi(X)$. What is the rank of the free group $\pi_1(X)$?
   (b) If X is a finite graph and $\tilde{X} \to X$ is an n-sheeted cover of X, what is the relationship between the Euler number of X and the Euler number of $\tilde{X}$?
3. (a) **(Centralizer).** Let G be a group. Let S be a subgroup of G (or, more generally, a subset). The *centralizer* of S is defined to be

$$C_G(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\}.$$

   Prove that $C_G(S)$ is a subgroup of G.
   (b) Explain the difference between the normalizer $N_G(S)$ of S and the centralizer $C_G(S)$ of S.
   (c) Prove that $C_G(S)$ is contained in $N_G(S)$, and that it is a normal subgroup.
   (d) Under what conditions on S will we have containment $S \subseteq C_G(S)$?
   (e) We have another name for the subgroup $C_G(G)$. What is it?
4. (a) Consider the transposition (12) in the symmetric group S_4. What is the normalizer of the subgroup $\langle(12)\rangle$ in S_4?
   (b) What is the normalizer of the subgroup $\langle a\rangle$ in the free group F_2 on a, b ?
5. **Definition (Abelian cover).** Let X be path-connected, locally path-connected, and semi-locally simply-connected. A cover $p\tilde{X} \to X$ is *abelian* if it is a regular cover with an abelian deck group.
   (a) Prove that X has an abelian cover U that is universal in the sense that it is a cover of every other abelian cover of X.
   (b) Verify that the cover $U \to X$ is unique up to isomorphism of covers.
   (c) What is U when $X = S^1 \vee S^1$?
6. **(Topology QR Exam, May 2017).** Let X be a connected CW-complex whose fundamental group is Σ_3, the group of all permutations on 3 elements.
   (a) How many isomorphism classes of objects are there in the category $\mathrm{Cov}_0(X)$ of connected covering spaces of X and continuous maps commuting with the covering map?
   (b) How many isomorphism classes of objects of $\mathrm{Cov}_0(X)$ have degree 2?
   (c) How many isomorphism classes of objects of $\mathrm{Cov}_0(X)$ are regular coverings?
7. **(Topology QR Exam, Jan 2018).** Let X be a graph with one vertex and two edges. Does there exist a connected covering $f : Y \to X$ which is regular and a connected covering $g : Z \to Y$ which is regular such that $fg : Z \to X$ is not a regular covering? Prove your answer.

8. Let Σ_g be a genus-g surface, and let $n \leq 2g$. Prove or disprove: Σ_g has a regular covering space with deck group $\mathbb{Z}^{2g}$.

9. Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a path-connected covering map, and let $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$. Prove that, if $[\gamma] \in \pi_1(X, x_0)$, then there is a point $\tilde{x_1} \in p^{-1}(x_0)$ with $p_*(\pi_1(\tilde{X}, \tilde{x_1})) = [\gamma]^{-1}H[\gamma]$.

10. Find free generating sets for the kernels of the following homomorphisms.

    (a)
    $$\begin{aligned} h : F_2 &\longrightarrow \mathbb{Z} \\ a &\longmapsto 1 \\ b &\longmapsto 0 \end{aligned}$$

    (b)
    $$\begin{aligned} h : F_2 &\longrightarrow \mathbb{Z} \\ a &\longmapsto 1 \\ b &\longmapsto 1 \end{aligned}$$

    (c)
    $$\begin{aligned} h : F_2 &\longrightarrow \mathbb{Z} \\ a &\longmapsto 1 \\ b &\longmapsto 2 \end{aligned}$$

11. **(Topology QR Exam, May 2020).** Describe a set of free generators of the subgroup of the free group on two generators a, b generated by b and all the conjugates of a^2, b^2, and $(ab)^3$. Is this a normal subgroup?

12. **(Topology QR Exam, Jan 2020).** Describe a set of free generators of the subgroup of the subgroup of the free group on two generators a, b generated by all conjugates of $aba^{-1}b^{-1}$.

13. Build an n-sheeted cover of $S^1 \vee S^1$, with vertices labelled $1, 2, \ldots, n$, such that the action of a on the vertices is given by the permutation (in cycle notation):
    (a) (123)
    (b) (12)(3)
    (c) (1)(2)(3)
    (d) (12)(34)

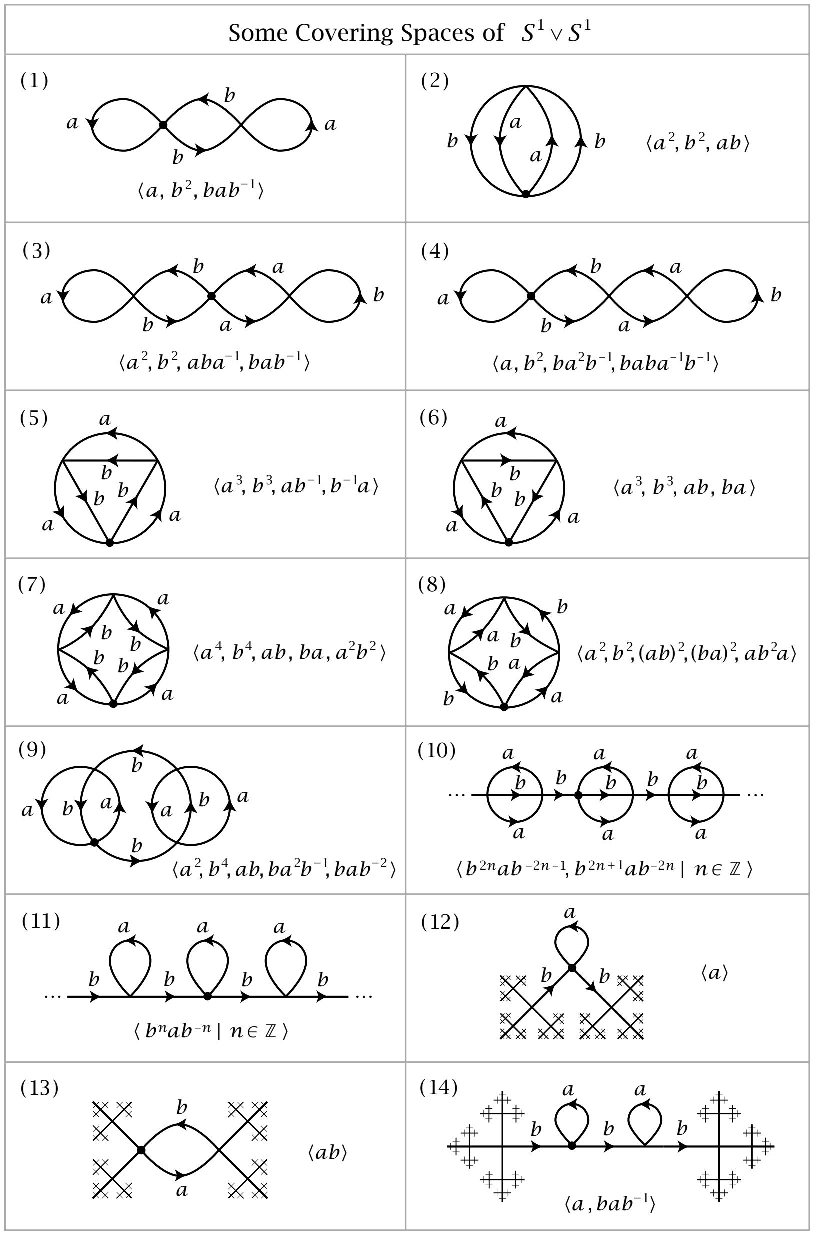
14. Let $p : \tilde{X} \to X$ be a connected, regular covering map, and let $x_0 \in X$. Let $G(\tilde{X})$ be the deck group. Build a bijection of sets between $G(\tilde{X})$ and the fibre $p^{-1}(x_0)$.

15. Suppose that $p : \tilde{X} \to X$ is a path-connected cover. Recall that we defined the cover to be *regular* if, for any $x \in X$, the group of Deck transformations of the cover acts transitively on the fibre $p^{-1}(x)$. Suppose that there is a single point $x_0 \in X$ such that the group of Deck transformations of the cover acts transitively on the fibre $p^{-1}(x_0)$. Prove that the cover is regular.

16. Let (X, x_0) be a based space, and let $p : \tilde{X} \to X$ be a cover.
    (a) Explain why the subgroup $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is independent of the choice of basepoint $\tilde{x_0} \in p^{-1}(x_0)$ if and only if H is normal.
    (b) Let $\gamma \in \pi_1(X, x_0)$. Show by example that the deck transformation of $\tilde{X}$ defined by γ may depend on the choice of basepoint $\tilde{x_0}$, even if H is normal.

(Note that the subgroup H, its normalizer $N(H)$, and the action of an element of $N(H)$ all depend on our choice of basepoint $\tilde{x_0}$).

17. Consider Hatcher's covers $\tilde{X}$ of $S^1 \vee S^1$ in the table below.



Each cover has a distinguished basepoint $\tilde{x_0}$ (marked by a black dot) in the preimage of the single vertex x_0 in $S^1 \vee S^1$. We identify $\pi_1(S^1 \vee S^1, x_0)$ with tthe free group F_2 on letters a, b. For each cover, answer the following questions.

(a) How many sheets is the cover?

(b) What is $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ as a subgroup of $\pi_1(S^1 \vee S^1, x_0) = F_2$? Give a free generating set.

(c) Is the cover $\tilde{X}$ regular?

(d) Describe the group of deck transformations of $\tilde{X}$.

(e) Number the vertices of $\tilde{X}$, and compute the permutations on the vertices defined by the actions of a, b, and ab in $\pi_1(S^1 \vee S^1, x_0)$.

(f) In our proof identifying the deck group of $\tilde{X}$ with $N(H)/H$, we defined an action of $N(H) \subseteq F_2$ by deck transformations. For each of the covers, for the elements $\gamma = a$ and $\gamma = b$ in F_2, determine whether γ is in $N(H)$, and, if so, describe the associated deck transformation of $\tilde{X}$.

18. Describe the universal cover of $S^1 \vee S^1$. Choose a cover from Hatcher's table, and explain/illustrate how it can be constructed as a quotient of the universal cover by a suitable choice of covering action.

19. Let $L(p, q)$ be the lens space with parameters p, q from Homework 7. Describe all (unbased) isomorphism classes of covering spaces of $L(p, q)$, and the system of all intermediate covering maps.

20. Let $p : \tilde{X} \to X$ be a connected cover, and let $x_0 \in X$.
    (a) Let $\tilde{x_1}, \tilde{x_2} \in p^{-1}(x_0)$. Prove that, if a deck transformation mapping $\tilde{x_1}$ to $\tilde{x_2}$ exists, then it is unique.
    (b) Suppose a deck transformation τ mapping $\tilde{x_1}$ to $\tilde{x_2}$ exists. Explain how to determine where τ maps an arbitrary point $\tilde{x} \in \tilde{X}$. *Hint:* Recall our construction from our proof of the existence of lifts.

21. Given a group G and a normal subgroup N , show that there exists a normal covering space $\tilde{X} \to X$ with $\pi_1(X) \cong G$, $\pi_1(\tilde{X}) \cong N$, and deck transformation group $G(\tilde{X}) \cong G/N$.

22. Let $S^{2k-1} \subseteq \mathbb{C}^k \cong \mathbb{R}^{2k}$ be the unit sphere. Define an action of $\mathbb{Z}/m\mathbb{Z}$ on S^{2k-1} by rotation, generated by the map $v \mapsto e^{2\pi i/m}v$. Compute the fundamental group of its orbit space.

23. Let G_1 act on X_1 and G_2 act on X_2 by covering space actions.
    (a) Define an action of $G_1 \times G_2$ on $X_1 \times X_1$ by
    $$(g_1, g_2) \cdot (x_1, x_2) = (g_1 \cdot x_1, g_2 \cdot x_2).$$
    Prove this is a covering space action.
    (b) Prove that $(X_1 \times X_2)/(G_1 \times G_2)$ is homeomorphic to $X_1/G_1 \times X_2/G_2$.

24. Consider the following actions of a group G on a space X. Determine which actions are covering space actions.
    (a) $X = S^1$, $G = \mathbb{Z}/2\mathbb{Z}$, and the generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by $180°$ rotation.
    (b) $X = S^2$, $G = \mathbb{Z}/2\mathbb{Z}$, and the generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by $180°$ rotation around the vertical axis.
    (c) $X = \mathbb{R}^n$, $G = \mathbb{R}$, and $r \in \mathbb{R}$ acts by
    $$r \cdot (x_1, x_2, x_3 \ldots, x_n) = (x_1 + r, x_2, x_3, \ldots, x_n).$$
    (d) $X = \mathbb{R}^n$, $G = \mathbb{Z}$, and $z \in \mathbb{Z}$ acts by
    $$z \cdot (x_1, x_2, x_3 \ldots, x_n) = (x_1 + z, x_2, x_3, \ldots, x_n).$$
    (e) $X = Y^n$ for some space Y, $G = S_n$, and $\sigma \in S_n$ acts by
    $$\sigma \cdot (y_1, y_2, \ldots, y_n) = (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}).$$

25. For each of the following actions: prove that they are covering actions, and identify the quotient (they are spaces we know by name!)
    (a) The group $\mathbb{Z}/2\mathbb{Z}$ acts on $S^1 \times I$. The generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by $(x, t) \mapsto (-x, t)$.
    (b) The group $\mathbb{Z}/2\mathbb{Z}$ acts on $S^1 \times I$. The generator $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by $(x, t) \mapsto (-x, 1 - t)$.

26. Find a covering space action of $\mathbb{Z}/2\mathbb{Z}$ on the torus so that the quotient is a Klein bottle.

27. Let G be a group with a covering space action on a path-connected, locally path-connected space $\tilde{X}$.
    (a) What can you say about the relationship between G and $\pi_1(X/G)$? (Note: we did not assume X is simply connected).
    (b) For a subgroup $H \subseteq G$, show that X/H is a cover of X/G.
    (c) Show that the cover $X/H \to X/G$ is normal if and only if H is a normal subgroup of G.
    (d) For subgroups H_1, H_2 of G, show that the covering space X/H_1 and X/H_2 of X/G are isomorphic if and only if H_1 and H_2 are conjugate subgroups of G.

28. Let $\tilde{X} \to X$ be a (not necessarily regular) cover, and let G be its group of deck transformations. Let $q : \tilde{X} \to \tilde{X}/G$ be the quotient map to the orbit space X/G. Show that there exists a map r making the following diagram commute.

$$\begin{array}{c} \tilde{X} \\ \downarrow q \\ \tilde{X}/G \\ \downarrow r \\ X \end{array} \qquad p : \tilde{X} \to X$$

29. Let $\{(C^i_*, d^i_*)\}_{i \in I}$ be a family of chain complexes.
    (a) How should we define the complex $\left(\bigoplus_i C^i_*, d_*\right)$?
    (b) Prove that (for a suitable solution to part (a)),

$$H_n\left(\bigoplus_i C^i_*\right) = \bigoplus_i H_n(C^i_*).$$

30. (a) Let $\{A_n\}_{n \in \mathbb{Z}_{\geq 0}}$ be a family of abelian groups. Construct a chain complex $\{(C_*, d_*)\}$ such that $H_n(C_*) = A_n$.
    (b) Let $\{A_n\}_{n \in \mathbb{Z}_{\geq 0}}$ be a family of finitely generated abelian groups. Construct a chain complex $\{(C_*, d_*)\}$ such that C_n is free abelian for all n, and $H_n(C_*) = A_n$.

31. Consider the abelian groups A and subgroups B given below. Compute the isomorphism type of the quotient A/B (in the sense of the structure theorem for finitely generated abelian groups).
    (a) $A = \mathbb{Z}^2$, B is the subgroup generated by $(2,3)$.
    (b) $A = \mathbb{Z}^2$, B is the subgroup generated by $(2,4)$.
    (c) $A = \mathbb{Z}^2$, B is the subgroup generated by $(1,1)$ and $(1,-1)$.
    (d) $A = \mathbb{Z}^2$, B is the subgroup generated by $(2,1)$ and $(7,4)$.
    (e) $A = \mathbb{Z}^2$, B is the subgroup generated by $(2,1)$ and $(1,3)$.

32. Put the following matrices into Smith normal form. Choose some matrices of your own and compute their Smith normal forms.

$$\begin{bmatrix} 1 & 3 & -2 \\ 5 & 2 & -1 \\ 6 & -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ -6 & 6 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

33. For a matrix A, what is the relationship between the Smith normal form of A and its transpose A^T?

34. Suppose A is a square matrix with entries in $\mathbb{Z}$.

(a) Suppose A is invertible over $\mathbb{Z}$. What are the possibilities for its Smith normal form?

(b) Suppose A is invertible over $\mathbb{Q}$ (but not necessarily over $\mathbb{Z}$). What are the possibilities for its Smith normal form?

35. Consider the quotient of $\mathbb{Z}^3$ by the subgroup generated by $\begin{bmatrix} 7 \\ 6 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Compute this group (in the sense of the structure theorem for finitely generated abelian groups).
Hint: Smith normal form.

36. Compute (without computer assistance) the homology of the following chain complexes.

$$\longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 4 & -8 \\ -4 & 8 \\ 4 & -8 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} -4 & -2 & 2 \\ 2 & 1 & -1 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow$$

$$\longrightarrow \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 2 & 1 & 2 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -2 \\ -1 & 1 & 2 \end{bmatrix}} \mathbb{Z}^3 \longrightarrow$$

$$\longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 2 & 2 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 1 & 4 & -2 \\ 0 & 3 & -3 \\ 1 & 1 & 1 \end{bmatrix}} \mathbb{Z}^3 \longrightarrow$$

37. For each of the following spaces, choose a Δ-complex structure, and compute the simplicial homology.

(a) a selection of finite graphs of your choosing

(b) a cylinder

(c) a Klein bottle

(d) the wedge sum of a closed 2-disk and a circle

(e) the disjoint union of a cylinder and a circle

38. **True or counterexample.** For each of the following statements: if the statement is true, write "True". If not, state a counterexample. No justification necessary.
Note: If the statement is false, you can receive partial credit for writing "False" without a counterexample.

(a) There is no 3-sheeted cover of $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$.

(b) There is no 4-sheeted cover of $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$.

(c) Every cover of a Mobius band is regular.

(d) Every cover of $S^1 \vee S^1$ is regular.

(e) Every covering map $p : \tilde{X} \to X$ is the quotient map to the orbit space of the action of the deck group $G(\tilde{X})$ on $\tilde{X}$.

(f) Let $\tilde{X} \to X$ be a connected covering space, and let τ be a deck map $\tilde{X} \to \tilde{X}$. If τ fixes a point, then τ is the identity.

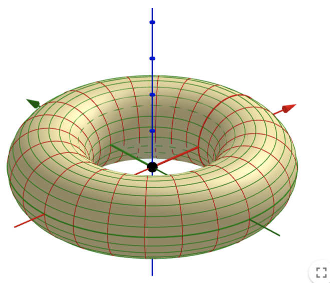
(g) Let X be a path-connected, locally path-connected, semi-locally simply-connected based space. Let H be a subgroup of $\pi_1(X)$, and let Y be a space with $\pi_1(Y) \cong H$. Then there exists a covering map $Y \to X$.

(h) Let $H_n(X)$ be the nth simplicial homology group of a Δ-complex X. Then the rank of $H_n(X)$ is at most the number of n-simplices in X.

(i) Let X be a 2-dimensional Δ-complex X. Then $H_1(X)$ is free abelian.

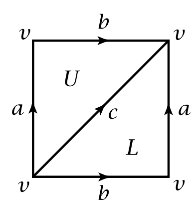
(j) Let X be a 2-dimensional Δ-complex X. Then $H_2(X)$ is free abelian.

39. Let T be a smoothly embedded torus in $\mathbb{R}^3$, as shown below. Compute the homology of the quotient space $\mathbb{R}^3/T$.



40. Let M be a Mobius band and let S be its boundary circle. Compute the homology of the quotient M/S using the long exact sequence of a pair. Verify your solution by a direct analysis of the homotopy type of the topological space M/S.

41. Let T be the torus with the following Δ-complex structure, and consider the subcomplex corresponding to the loop a.



Compute the relative homology groups $H_*(T, a)$ ...

(a) ... using the long exact sequence of a pair.

(b) ... by computing the quotient of simplicial chain complexes $C_*(T)/C_*(a)$ and taking homology.

42. Let X be a CW complex, and let X^k denote its k-skeleton.

(a) Show that the quotient X^k/X^{k-1} is homotopy equivalent to a wedge of k-dimensional spheres, one for each k-cell of X.

(b) What is $H_*(X^k, X^{k-1})$?

(c) Verify your answer in the case that X is a Δ-complex, by describing the quotient of the simplicial chain complexes $C_*(X^k)/C_*(X^{k-1})$ and computing its homology.

43. Let G be a group acting on a space X. Show that, for each n, there is an induced group action of G on $H_n(X)$.

44. (a) Let $A \subseteq X$. Prove or find a counterexample: for each n,

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

(b) Let $A \subseteq X$ and suppose A is a retract of X. Prove or find a counterexample: for each n,

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

45. Must the following maps be nullhomotopic? Give a proof or prove a counterexample.
    (a) $f : S^2 \to S^1 \times S^1$
    (b) $g : S^1 \times S^1 \to S^2$

46. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an integer matrix.
    (a) Show that the map $A : \mathbb{R}^2 \to \mathbb{R}^2$ induces a well-defined map on the orbit space $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$.
    (b) Recall that $\mathbb{R}^2/\mathbb{Z}^2$ is the torus T. Compute the map induced by A on $H_1(T)$.