Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (n-sphere) (we sometimes view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n\geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

Practice problems

1. Let X be a topological space, and let $f, g : X \to S^n$ be two maps with the property that the points $f(x)$ and $g(x)$ are never antipodal for any $x \in X$. Prove that f and g are homotopic.

2. Suppose that a map $f : S^1 \to S^1$ is nullhomotopic. Show that f has a fixed point, and maps at least one point x to its antipode $-x$.

3. Let X and Y be topological spaces.
   (a) Suppose that Y is contractible. Prove that any two maps from X to Y are homotopic.
   (b) Suppose that X is contractible and Y is path-connected. Show that any two maps from X to Y are homotopic.

4. (a) Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y .
   (b) Show that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y .

5. A topological space G with a group structure is called a *topological group* if the group multiplication map and inverse map

$$G \times G \longrightarrow G \qquad (g, h) \longmapsto g \cdot h \qquad\qquad G \longrightarrow G \qquad g \longmapsto g^{-1}$$

are continuous. Suppose G is a path-connected topological group. Show that, for each $g_0 \in G$, the left multiplication map

$$\ell_{g_0} : G \longrightarrow G \qquad g \longmapsto g_0 \cdot g$$

is homotopic to the identity.

6. Let X be a CW complex. Show that any finite collection of cells in X are contained in a finite subcomplex.

7. Give an example (with proof) of a topological space that does not admit a CW complex structure.

8. Let X be a CW complex with n_d cells in dimension d, and let Y be a CW complex with m_d cells in dimension d. We described a natural CW complex structure on $X \times Y$. Count its cells in dimension d.

9. (a) Let X be a CW complex, and let $q : \bigsqcup_{n,\alpha} D^n_\alpha \to X$ be the map defined by the characteristic maps. Show that q is a quotient map.
   (b) Let X be a CW complex and Y any space. Use part a to show that a map $\phi : X \to Y$ is continuous if and only if its restriction to the closure of every cell of X is continuous.

10. In this problem, we will show that CW complexes may not be metrizable. Let X be a 1-dimensional CW complex (that is, a graph).

(a) Let $x_1, x_2, \ldots,$ be a sequence of points in X, each in distinct edges. Explain why $(x_n)_n$ cannot converge.

(b) Suppose that v is a vertex in X with infinitely many edges incident on v. Explain why the weak topology on X cannot be metrizable.

11. **Definition (Isomorphism).** Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is called an *isomorphism* if there is a morphism $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$. In this case, we write $g = f^{-1}$ and call g the *inverse* of f.

(a) Suppose a morphism $f : X \to Y$ in a category $\mathscr{C}$ has an inverse g. Verify that the inverse is unique (so calling g "the" inverse is justified.)

(b) Show that an isomorphism is both monic and epic.

(c) Show that a map can be monic and epic, but not an isomorphism. *Hint:* Consider a category with only two objects.

(d) Let $f : X \to Y$ be a morphism in a category $\mathscr{C}$. Suppose there were morpihsms $g, h : Y \to X$ such that $f \circ g = id_Y$ and $g \circ h = id_X$. Show that $g = h$, and conclude that f is an isomorphism.

(e) Let $f : X \to Y$ be a map of topological spaces. Suppose there exists maps $g, h : Y \to X$ so that $f \circ g$ and $h \circ f$ are both homotopic to identity maps. Prove that f is a homotopy equivalence.

12. Show that a map homotopic to a homotopy equivalence is itself a homotopy equivalence.

13. Let $\underline{\text{Set}}$ be the category of sets, and for a set A let $\text{Hom}_{\underline{\text{Set}}}(-, A)$ denote the associated contravariant hom functor

$$\begin{aligned} \text{Hom}_{\underline{\text{Set}}}(-, A) : \mathscr{C} &\longrightarrow \underline{\text{Set}} \\ B &\longmapsto \text{Hom}_{\underline{\text{Set}}}(B, A) \\ [f : B \to C] &\longmapsto \left[f^* : \begin{array}{cc} \text{Hom}_{\underline{\text{Set}}}(C, A) & \to \text{Hom}_{\underline{\text{Set}}}(B, A)] \\ \phi & \mapsto \phi \circ f \end{array} \right] \end{aligned}$$

Prove that, if f is surjective, then f^* is injective.

14. Let $\underline{\text{Top}}_*$ be the category of based topological spacces and based maps. What is the coproduct of based spaces X and Y, along with the two associated maps? Prove your answer.

15. Let $\underline{\text{Top}}$ be the category of topological spaces, and let $P : \underline{\text{Top}} \to \underline{\text{Set}}$ be the map that takes a topological space X to its set of path components. Explain how to define P on morphisms to make it a functor, and verify that it is well-defined and functorial.

16. Recall that we defined a forgetful map

$$\begin{aligned} \underline{\text{Top}}_* &\longrightarrow \underline{\text{Top}} \\ (X, x_0) &\longmapsto X \\ f &\longmapsto f \end{aligned}$$

What is the "free functor" associated to this "forgetful functor"? For a topological space X, determine what universal property the "free based space on X" should satisfy, and describe what this topological space $F(X)$ (along with its basepoint, and distinguished map $X \to F(X)$) should be.

17. **Definition (Initial objects, terminal objects, zero objects).** An *initial object* I in a category $\mathscr{C}$, if it exists, is an object with the property that for any $X \in \mathscr{C}$, there is a unique morphism in $\mathscr{C}$ from I to X. Dually, an object T is a *terminal object* if for every $X \in \mathscr{C}$ there is a unique morphism $X \to T$. If an object is both initial and terminal, it is called a *zero object.*

(a) Show that, if an initial (or terminal, or zero) object exists in a category $\mathscr{C}$, it is unique up to unique isomorphism.

(b) Determine whether initial, terminal, or zero objects exists, and what they are, in the categories $\underline{\text{Set}}$, $\underline{\text{Ab}}$, $\underline{\text{Grp}}$, $\underline{\text{Top}}$, and $\underline{\text{Top}}_*$.

(c) Let $\mathscr{C}$ be a category and I an initial object in $\mathscr{C}$. Prove or disprove: If $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor, then $F(I)$ is an initial object in $\mathscr{D}$.

18. Let $F(S)$ denote the free group on the set S. Suppose that S_1 and S_2 are finite sets. Show that, if S_1 and S_2 have different cardinalities, then $F(S_1)$ and $F(S_2)$ are not isomorphic.
Hint: Show that $\text{Hom}_{\underline{\text{Grp}}}(F(S), \mathbb{Q})$ has the structure of a $\mathbb{Q}$-vector space, and compute its dimension.

19. (a) Let G be a group and $[G, G]$ is commutator subgroup. Let $\phi : G \to G$ be an automorphism. Prove that $\phi([G, G]) \subseteq [G, G]$. (This means that $[G, G]$ is a *characteristic* subgroup.)

(b) Show that an automorphism $\phi : G \to G$ induces an automorphism $\tilde{\phi} : G^{ab} \to G^{ab}$.

(c) Let $\text{Aut}(G)$ denote the group of automorphisms of G. Prove that the induced map

$$\Phi : \text{Aut}(G) \longrightarrow \text{Aut}(G^{ab})$$
$$\phi \longmapsto \tilde{\phi}$$

is a group homomorphism.

20. (a) Let X be a space, and let x_0 and x_1 be two points in the same path component of X. Given a path h from x_0 to x_1, let β_h be the associated change-of-basepoint map

$$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0).$$

Choose a (possibly different) path g from x_1 to x_0. Show that the automorphism

$$\beta_h \circ \beta_g : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is given by conjugation by an element of $\pi_1(X, x_0)$ (which one?). Such automorphisms are called *inner automorphisms.*

(b) Explain the sense in which elements of $\pi_1(X, x_0)$ depend on the choice of basepoint, but conjugacy classes of elements in $\pi_1(X, x_0)$ do not.

(c) Suppose that $\pi_1(X, x_0)$ is abelian. Show that the isomorphism

$$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$$

is independent of choice of path h.

(d) Suppose X is path-connected and $\pi_1(X, x_0)$ is abelian. Explain the sense in which a loop in X gives a well-defined element of $\pi_1(X, x)$ with respect to any basepoint x.

21. We showed that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ is generated by the class of a loop α around a meridian circle and the class of a loop a longitudinal circle, as in Figure 1. Construct an explicit homotopy from $\alpha \cdot \beta$ to $\beta \cdot \alpha$.

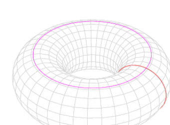


Figure 1: The loops α and β in Σ_1

22. (a) Suppose that $p_X : \tilde{X} \to X$ is a cover of a space X, and $p_Y : \tilde{Y} \to Y$ is a cover of a space Y. Show that $p_X \times p_Y : \tilde{X} \times \tilde{Y} \to X \times Y$ is a covering map.
    (b) Conclude that $\mathbb{R}^n$ is a cover of the n-torus.

23. Suppose that $p_X : \tilde{X} \to X$ is a cover of a space X.
    (a) Show that, if X is locally Euclidean then so is $\tilde{X}$.
    (b) Show that, if X is Hausdorff, then so is $\tilde{X}$.

    With this problem, we are most of the way to showing that, if X is a topological manifold, then so is $\tilde{X}$. We will need an extra condition on $\tilde{X}$, however, to ensure it is second-countable.

24. Let $f : X \to S^1$ be a continuous map. Show that, if f is nullhomotopic, then it factors through the covering map $p : \mathbb{R} \to S^1$.

25. Prove that A space X is simply-connected if and only if there is a unique homotopy class of paths connecting any two points in X .

26. Let $X \subseteq D^2$ be a subspace, and let $f : X \to X$ be a map without fixed points. Show that X is not a retract of D^2.

27. Show that there is no retraction from a Möbius band to its boundary.

28. Let $S^1 \times I$ be the cylinder, and suppose that $f : S^1 \times I \to X$ is a map that is constant on $S^1 \times \{1\}$. Show that the map f_* induced by f on fundamental group is trivial.

29. (a) Let $C_n \subseteq \mathbb{R}^2$ be the circle of radius n and center $(n, 0)$. Let $X = \cup_n C_n$. Show that $\pi_1(X)$ is the free group on countably many generators, with n^{th} generator a loop around C_n.
    (b) Show that X is not homeomorphic to the infinite wedge $\bigvee_{n\in\mathbb{N}} S^1$.
    Hint: Show that the point $(0, 0)$ in X has a countable neighbouhood basis, but the wedge point of $\bigvee_{n\in\mathbb{N}} S^1$ does not.

30. Let G and H be groups. Prove that $(G * H)^{ab} = G^{ab} \oplus H^{ab}$.

31. Let G be a group with presentation $\langle S \mid R\rangle$. Explain how to construct a 2-dimensional CW complex with fundamental group isomorphic to G.

32. Describe the construction of a CW complex with fundamental group is

$$\mathrm{SL}_2(\mathbb{Z}) \cong \langle a, b \mid abab^{-1}a^{-1}b^{-1}, (aba)^4\rangle.$$

33. Let X be a finite CW complex. Explain why $\pi_1(X)$ must be a finitely presented group.

34. (a) State the Cellular Approximation Theorem.
    (b) Let $d \leq n - 1$. Prove that every map from a d-dimensionl CW complex X to the n-sphere S^n is nullhomotopic.

35. **True or counterexample.** For each of the following statements: if the statement is true, write "True". If not, state a counterexample. No justification necessary.
    Note: If the statement is false, you can receive partial credit for writing "False" without a counterexample.
    (a) Any contractible space is path-connected.
    (b) Any subspace of a contractible space is contractible.
    (c) Any quotient of a contractible space is contractible.
    (d) The product of two contractible spaces is contractible.
    (e) For any topological space X, any two maps $X \to S^\infty$ are homotopic.

(f) For any topological space X, any two maps $\mathbb{R}^n \to X$ are homotopic.

(g) Let X be a space and A a subspace. If A is a retract of X, then A and X are homotopy equivalent. (Note the distinction between *retract* and *deformation retract*).

(h) A CW complex X is compact if and only if it is finite.

(i) Any compact subset of a CW complex is closed.

(j) Let F_S be the free group on a set S. Then every group isomorphism $F_S \to F_S$ is induced by a permutation of S.

(k) There exists no non-abelian group with abelianization $\mathbb{Z}$.

(l) Suppose a group G has a presentation with a single generator. Then G is abelian.

(m) If G is a finitely presented group, then its abelianization G^{ab} is finitely presented.

(n) Every topological space is homeomorphic to the topological disjoint union of its connected components.

(o) There does not exist a retraction from $\mathbb{R}^2$ to $\mathbb{R}^2 \setminus \{0\}$.

(p) There does not exist a retraction from the torus $S^1 \times S^1$ to its meridian $S^1 \times \{(1,0)\}$.

(q) Let $X \subseteq Y$ be a subspace, and $x_0 \in X$. Then $\pi_1(X, x_0)$ is a subgroup of $\pi_1(Y, x_0)$.

(r) Suppose that X is a union of open, contractible subsets. Then $\pi_1(X) = 0$.

(s) Any presentation of a finite group must be finite.

(t) If G is a finitely presented group, then there exists a compact Hausdorff space with fundamental group isomorphic to G.

(u) If X is a connected graph, then $\pi_1(X)$ is a free group.

36. Compute the fundamental groups of the following spaces, and give presentations. Draw or describe the generators as loops in the space. Recall some tools we have developed:

- fundamental group is a homotopy invariant
- fundamental groups of Cartesian products and wedge sums
- the quotient of a CW complex by a contractible subcomplex is a homotopy equivalence
- Van Kampen theorem
- our results from Homework #4 on the effect on π_1 of gluing in disks along their boundaries
- our results from Homework #4 on the fundamental groups of CW complexes

(a) The product of $\mathbb{RP}^2$ and $\mathbb{CP}^2$

(b) The wedge sum of a cylinder and a Möbius band

Figure 2: The wedge sum of a cylinder and a Möbius band

(c) A genus-2 surface with two disks glued in, as in Figure 3

(d) A Klein bottle

(e) Two Möbius bands glued by the identity map along their boundaries.

(f) Each of graphs in Figure 4.

(g) The CW complex shown in Figure 5.

(h) The plane $\mathbb{R}^2$ with n punctures

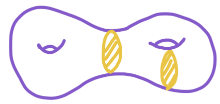


Figure 3: Two disks glued into a closed genus-2 surface

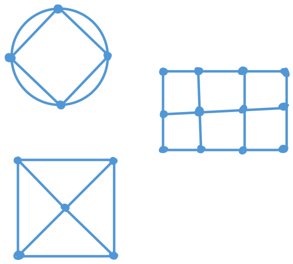


Figure 4: Three connected graphs

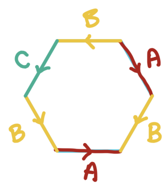


Figure 5: A 2-dimensional CW complex

(i) The 2-sphere S^2 with n punctures
(j) $\mathbb{R}^3$ after deleting n lines through the origin
(k) The torus with n punctures
(l) $\mathbb{RP}^2$ with a puncture
(m) The quotient of the torus obtained by choosing an embedded disk D^2, and identifying its boundary S^1 to a single point

37. **(QR Exam, January 2016).** Let K be the complete graph on 4 letters, ie, K has 4 vertices, and there is a unique edge connecting each pair of distinct vertices.
(a) Calculate $\pi_1(K)$.
(b) Show that Σ_2 is not a deformation retract of any space homotopy equivalent to K.

38. **(QR Exam, September 2016).** Let

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \quad D = \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

Consider in the torus $T = S^1 \times S^1$ the homeomorph of the open disk

$$U = \{(x, y) \mid Re(x),\ Re(y) > 1/2\}.$$

Consider a homeomorphism $h : S^1 \to \partial U$ where ∂U is the boundary of the closure of U in T, and let $f : S^1 \to \partial U$ be the map given by $f(z) = h(z^2)$. Now let X be the quotient of

$$D \sqcup (T \setminus U)$$

by the smallest equivalence relation which has $z \sim f(z)$ for $z \in S^1 \subseteq D$. Find $\pi_1(X)$ in terms of generators and relations.

39. **(QR Exam, January 2017).** Prove that the usual inclusions $\mathbb{C}\mathrm{P}^0 \subseteq \mathbb{C}\mathrm{P}^1 \subseteq \cdots \subseteq \mathbb{C}\mathrm{P}^n$ define a CW filtration on $\mathbb{C}\mathrm{P}^n$.

40. **(QR Exam, January 2019).** Let S^1 be the unit circle in $\mathbb{C}$. Let Y by the space obtained from $S^1 \times [0,1] \times \{0,1\}$ (with the product topology, where each factor has the standard topology) by identifying $(z,0,0) \sim (z^3,0,1)$ and $(z,1,1) \sim (z^2,1,0)$. Calculate $\pi_1(Y)$.

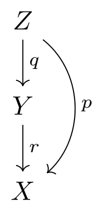
41. Let X be a Hausdorff space. Prove that any cover $\tilde{X}$ of X must be Hausdorrff.
Remark: The converse is not true!

42. Let $p : \tilde{X} \to X$ be a covering map with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $\tilde{X}$ is compact Hausdorff if and only if X is compact Hausdorff.

43. Let $p : \tilde{X} \to X$ be a covering space. Prove that X has a basis of open subsets that are evenly covered by p.

44. For each of the following maps: prove that the map is a covering map, or prove that it is not a covering map.
    (a) The defining quotient map $S^n \to \mathbb{R}\mathrm{P}^n$.
    (b) The defining quotient map $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$.

45. Let X, Y, Z be path-connected, locally path-connected, and semi-locally simply-connected spaces. Let p, q, r be continuous maps with $p = r \circ q$. Show that, if p and q are covering maps, then so is r.



46. Let $p : \tilde{X} \to X$ be a covering space, and let $\{U_\alpha\}$ be an open cover of X with the property that each open subset U_α is evenly covered by p. Let $f : I \to X$ be a continuous path. Prove that there is a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each i, $f|_{[t_i, t_{i+1}]} \subseteq U_i$ for some $U_i \in \{U_\alpha\}$.

47. (a) Suppose that X is a graph. Show that, if X is simply connected, then X is contractible.
    (b) Let X be a graph. Conclude that the universal cover of X is contractible.
    (c) Let X be a graph, and let L be a connected space with a finite fundamental group. Prove that every continuous map from L to X is nullhomotopic.

48. Let $p : \tilde{Y} \to Y$ be a covering map, and let $f : X \to Y$ be any map.
    (a) Define a *lift* of f to $\tilde{Y}$, and give sufficient conditions for a lift $\tilde{f}$ to exist.
    (b) Explain how we constructed the lift $\tilde{f}$.
    (c) In what sense is the lift unique? Show by example how, when X is not connected, the lift $\tilde{f}$ need not satisfy our uniqueness condition.

49. Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering map. Prove that the induced map $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ is injective.

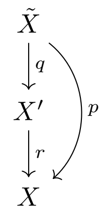
50. Let $n \geq 2$. Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering map, and let $f : (S^n, s_0) \to (X, x_0)$ be a based map. Prove that f lifts to a unique based map $f : (S^n, s_0) \to (\tilde{X}, \tilde{x_0})$.

51. **(Topology QR Exam, Aug 2021).** Let $n \geq 0$. Let $\mathbb{C}\mathrm{P}^n$ denote complex projective n-space, and let $x_0 \in \mathbb{C}\mathrm{P}^n$ be a fixed basepoint. Let S^1 denote the circle, and let $y_0 \in S^1$ be a fixed basepoint. Give an explicit proof that every based map

$$f : (\mathbb{C}\mathrm{P}^n, x_0) \to (S^1, y_0)$$

is nullhomotopic via a *basepoint-preserving homotopy*, i.e., a homotopy f_t satisfying $f_t(x_0) = y_0$ for all t.

52. Let X be a path-connected, locally path-connected, semi-locally simply connected space.
    (a) Show that the universal cover $p : \tilde{X} \to X$ of X is universal in the following sense: Given any covering map $r : X' \to X$, there is covering map $q : \tilde{X} \to X'$ making the following diagram commute.



    (b) Is the covering map $q : \tilde{X} \to X'$ unique?

53. (a) Let $p : \tilde{X} \to X$ be a covering map. Explain why, if $\tilde{X}$ is simply connected, then it must be the universal cover (or, more precisely, it must be isomorphic as a cover to the universal cover).
    (b) Read Tai-Danae Bradley's 'recipe' for constructing the universal cover of a wedge sum, and describe the procedure in your own words.
    `https://www.math3ma.com/blog/a-recipe-for-the-universal-cover-of-x-y`
    (c) Give explicit descriptions (possibly by picture) of the universal covers of the following spaces.
        (i) $S^1 \times S^1 \times S^1$
        (ii) $S^1 \vee S^1 \vee S^1$
        (iii) $S^2 \vee S^1$
        (iv) $\mathbb{R}\mathrm{P}^n$, $n \geq 2$
        (v) $\mathbb{C}\mathrm{P}^n$

54. Suppose that X is not semi-locally simply connected. Explain why X cannot have a universal cover.

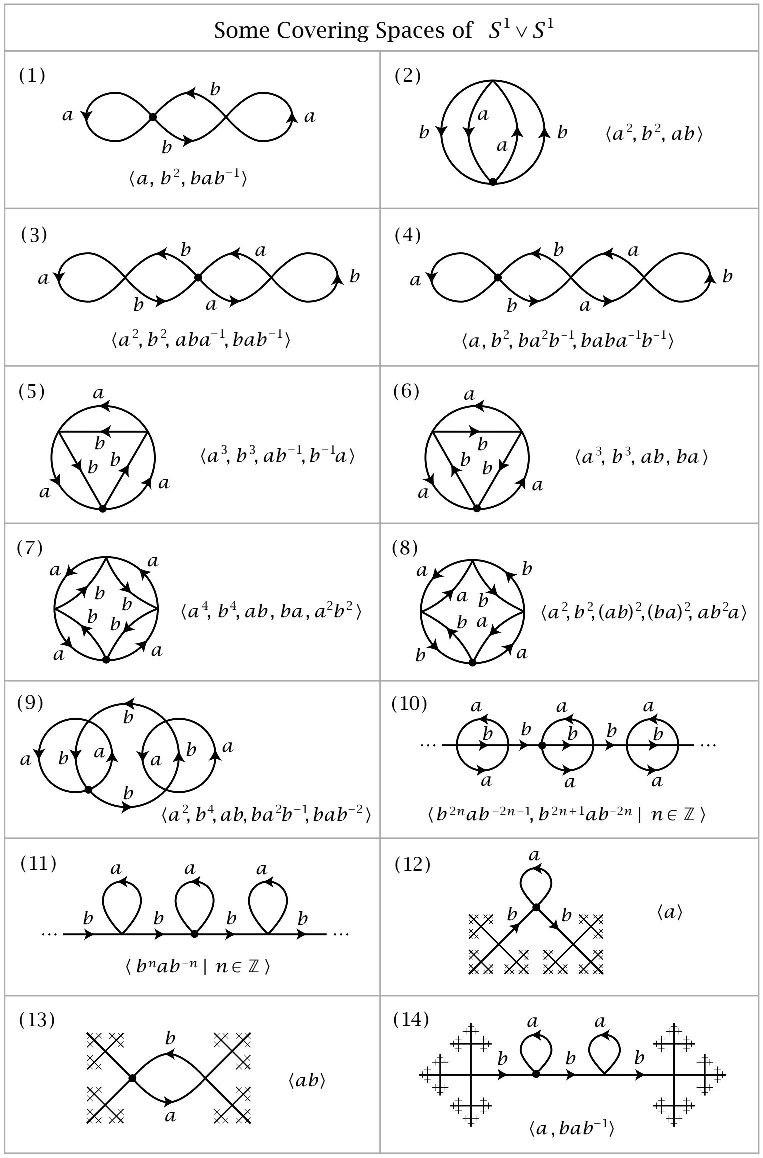
55. **(Topology QR Exam, Jan 2021).** Let $X = \mathbb{R}\mathrm{P}^3$ and let $Y = S^1 \vee S^1$.
    (a) Are all maps $f : X \to Y$ null-homotopic?
    (b) Are all maps $f : Y \to X$ null-homotopic?
    For each of the above, give a proof if the answer is "yes" and give an example if the answer is "no".

56. **(Topology QR Exam, Aug 2020).** For each of the following cases, determine if there exists a covering space $f : X \to Y$. (If yes, then construct it; if not, then give a proof).
    (a) X is homotopy equivalent to $S^1 \times S^1$ and Y is homotopy equivalent to $S^1 \vee S^1$.
    (b) X is homotopy equivalent to $S^1 \vee S^1$ and Y is homotopy equivalent to $S^1 \times S^1$.

57. Let X and Y be path-connected, locally path-connected spaces with universal covers $p_X : \tilde{X} \to X$ and $p_Y : \tilde{Y} \to Y$. Show that a continuous map $f : X \to Y$ lifts to a continuous map $f' : \tilde{X} \to \tilde{Y}$ such that $p_Y \circ f' = f \circ p_X$. To what extent is f' unique?

58. Consider Hatcher's covers $\tilde{X}$ of $S^1 \vee S^1$ in the table below. Each cover has a distinguished basepoint $\tilde{x_0}$ (marked by a black dot) in the preimage of the single vertex x_0 in $S^1 \vee S^1$. We identify $\pi_1(S^1 \vee S^1, x_0)$ with tthe free group F_2 on letters a, b. For each cover, answer the following questions.
    (a) How many sheets is the cover?

(b) What is $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ as a subgroup of $\pi_1(S^1 \vee S^1, x_0) = F_2$? Give a free generating set.



59. **True or counterexample.** For each of the following statements: if the statement is true, write "True". If not, state a counterexample. No justification necessary.
Note: If the statement is false, you can receive partial credit for writing "False" without a counterexample.

(a) If $p : \tilde{X} \to X$ is a covering space map, then p cannot be nullhomotopic.

(b) If X is simply-connected, then X is semi-locally simply-connected.

(c) If X is connected, then X is semi-locally simply-connected.

(d) If X is locally simply-connected and semi-locally simply-connected, then X is simply-connected

(e) Let $p : \tilde{Y} \to Y$ be a covering map, and let $f : X \to Y$ be a map such that a lift $\tilde{f} : X \to \tilde{Y}$ exists. Then the lift $\tilde{f}$ is the unique map lifting f.