Name: ______________________________ Score (Out of 5 points):

1. (a) (2 points) Prove that a space is contractible if and only if the identity map is nullhomotopic.

Solution. By definition, a space is contractible iff it is homotopy equivalent to a point $*$. Suppose Y is contractible. Then the (neccesarily constant) map $f : Y \to \{*\}$ has some homotopy inverse $g : \{*\} \to Y$. The map $g \circ f : Y \to Y$ is the constant map at $g(*)$, and is, by definition of homotopy equivalence, homotopic to id_Y. Thus id_Y is nullhomotopic.

Now suppose id_Y is homotopic to a constant map $Y \to Y$ at some point $y_0 \in Y$. Then consider the constant map $f : Y \to \{*\}$ and the map $g : \{*\} \to Y$ that maps $*$ to y_0. Then $f \circ g = id_{\{*\}}$, and $g \circ f$ is the constant map at y_0, which is homotopic to id_Y by assumption. Thus f and g are homotopy inverses, and Y is contractible.

(b) (3 points) Prove that a retract of a contractible space is contractible.

Solution. Let X be a space and $A \subseteq X$. Suppose A is a *retract* of X, that is, suppose that there exists a map $r : X \to A$ such that $r|_A = id_A$.

Since X is contractible by assumption, by part (a) there must be a homotopy H_t from id_X to the constant map at some point $x_0 \in X$. Concretely, $H_0(x) = x$ for all $x \in X$, and $H_1(x) = x_0$ for all $x \in X$. So consider the homotopy

$$\begin{aligned} F : A \times I &\to A \\ F_t(a) &= r(H_t(a)) \end{aligned}$$

Then F is continuous because it is the composite of continuous functions. Moreover,

$$F_0(a) = r(H_0(a)) = r(a) = a$$

so $F_0 = id_A$, and

$$F_1(a) = r(H_1(a)) = r(x_0)$$

so F_1 is the constant map at $r(x_0) \in A$.

Thus id_A is nullhomotopic and, by part (a), A is contractible.