Name: ______________________________ Score (Out of 8 points):

1. A *terminal object* in a category $\mathscr{C}$ (if it exists) is an object T that satisfies the following universal property: for every object X in $\mathscr{C}$, there exists a unique morphism $f : X \to T$.

   (a) (3 points) Suppose that a terminal object T exists in a certain category $\mathscr{C}$. Prove that it is "unique up to unique isomorphism".

   Solution. Suppose T and T' are two objects of $\mathscr{C}$ satisfying the universal property. Then there is a unique morphism $f : T \to T'$ (by the universal property for T') and a unique morphism $g : T' \to T$ (by the universal property for T). We will prove that f, g are two-sided inverses, and thus isomorphisms. They are unique isomorphisms in the sense that they are the only morphisms between T and T'.

   Note that, by the definition of a category, there must exist an identity morphism $id_T : T \to T$. By the universal property of T, this must be the only morphism $T \to T$. Similarly the identity morphism $id_{T'} : T' \to T'$ is the only morphism $T' \to T'$.

   It follows that the maps $g \circ f : T \to T$ and $f \circ g : T' \to T'$ must be equal to id_T and $id_{T'}$, respectively. Thus f and g are isomorphisms. The objectst T and T' are canonically isomorphic, as claimed.

   (b) (2 points) Suppose T is a terminal object in a category $\mathscr{C}$. Prove that, for any object X and morphism $f : T \to X$, the morphism f is monic.

   Solution. Recall that a morphism $f : T \to X$ is *monic* if, whenever $f \circ g = f \circ g'$, then $g = g'$. So suppose that, for some object Y and morphisms $g, g' : Y \to T$, we have $f \circ g = f \circ g'$. But the universal property of T implies that there is a unique morphism $Y \to T$, so necessarily $g = g'$.

(c) (3 points) What are the terminal objects in the following categories? State a terminal object if one exists, otherwise write `"Does not exist"`. No justification needed.

Let $\mathscr{C}$ be the category ...

Set of sets and all functions:	A singleton set $\{*\}$
Grp of groups and group homomorphisms:	The zero group
Fld of fields and field homomorphisms:	Does not exist
Top of topological spaces and continuous maps:	A one-point space $\{*\}$
Top_* of based topological spaces and based continuous maps:	A one-point based space $(\{*\}, *)$
hTop of topological spaces and homotopy classes of continuous maps:	A one-point space $\{*\}$ (or any contractible space)