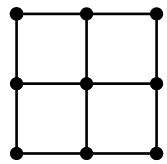
Name: ______________________ Score (Out of 4 points):

1. For each of the following spaces X,

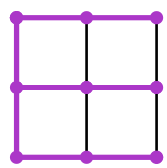
   - determine a presentation for the fundamental group
   - describe loops in the space representing the generators, either by written description or by a picture. Please describe loops in (a space homeomorphic to) X, not just in a homotopy-equivalent space.

   You do not need to give rigorous proofs, but briefly explain the steps in your reasoning.

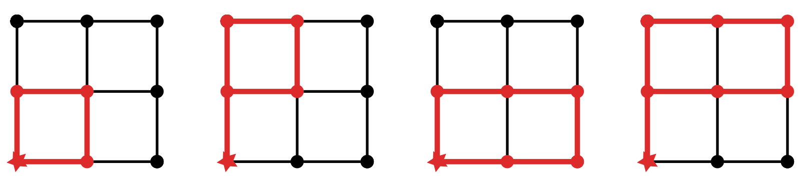
   (a) (2 points) X is the graph below.



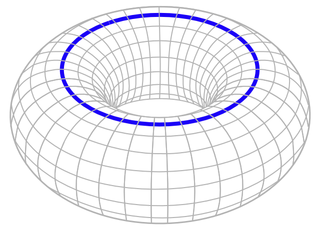
Sample solution. We can construct a homotopy equivalence between X and a wedge of four circles by taking the quotient by collapsing a choice of maximal tree T, shown below in purple. (Each solution will depend on the choice of tree).



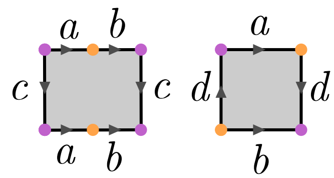
We obtain free generators for π_1 by taking a lift of each circle in X/T, based at the vertex marked with a star. In other words, for each of the four edges not in the maximal tree, we choose a path from the basepoint through the tree that traverses the edge exactly once and returns to the basepoint along a path through the tree. So $\pi_1(X) \cong \langle a, b, c, d \mid \rangle$. A choice of free generating set a, b, c, d is given by the homotopy classes of the four loops in the figure below. (Either choice of orientation on each loop is valid).



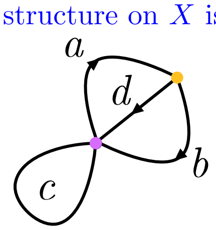
(b) (2 points) X is obtained by gluing a Mobius band by its boundary homeomorphically onto a longitudinal loop of a torus (shown below).



Sample solution. Observe that the space X admits the following CW complex structure. The torus is the quotient of the square on the left, and the Mobius strip the quotient of the square on the right. The boundary of the Mobius band is the loop ab.



The 1-skeleton of this CW complex structure on X is below.



If we choose the edge a to be a maximal tree, then we find a choice of free generators of its fundamental group (based at the purple vertex)

$$A = ad, \qquad B = ab, \qquad C = c$$

We view these same three words as loops in X; they generate its fundamental group. The two disks are glued along the words

$$abcb^{-1}a^{-1}c^{-1} = BCB^{-1}C^{-1} \qquad \text{and} \qquad adb^{-1}d = adb^{-1}a^{-1}ad = AB^{-1}A$$

Thus we obtain the presentation $\pi_1(X) = \langle A, B, C \mid BCB^{-1}C^{-1}, AB^{-1}A\rangle$.

Alternate solution outline. Alternatively, we can apply van Kampen. Let U_1 be a neighbourhood of the torus and U_2 a neighbourhood of the Klein bottle, chosen to be sufficiently small that they deformation retract back to the torus and Klein bottle, respectively. So $\pi_1(U_1) = \mathbb{Z}^2 = \langle B, C \mid BCB^{-1}C^{-1}\rangle$ and $\pi_1(U_2) = \mathbb{Z} = \langle A \mid \rangle$.
Their free product is $\pi_1(U_1) * \pi_1(U_2) = \langle A, B, C \mid BCB^{-1}C^{-1}\rangle$. The intersection $U_1 \cap U_2$ is homotopy equivalent to the circle. Its fundamental group's generator includes into $\pi_1(U_1)$ as the longitudinal loop B and into $\pi_1(U_2)$ as the boundary loop A^2. We obtain the relation $A^2 = B$ in the amalgamated free product. Thus $\pi_1(X) = \langle A, B, C \mid BCB^{-1}C^{-1}, A^2B^{-1}\rangle$. This agrees with the presentation above (up to conjugating the second relator by A).