Final Exam

Math 490
13 December 2024
Jenny Wilson

Name: ____

Instructions: This exam has 6 questions for a total of 50 points.

Each student may bring in one double-sided ($8\frac{1}{2}$" $\times$ 11") sheet of notes, which they must have either hand-written or typed (in font size at least 12) themselves.

The exam is closed-book. No books, additional notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any (non-optional) results proved on the worksheets, on a quiz, or on the homeworks without proof.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	11	
2	5	
3	4	
4	24	
5	3	
6	3	
Total:	50	

1. (11 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let (X, d) be a metric space. Then the function $\tilde{d}(x, y) = \frac{1}{10}d(x, y)$ defines a valid new metric on X.

   (b) Let X and Y be homeomorphic metric spaces. If X is bounded, then so is Y.

   (c) Let X be a **finite** topological space. If X has the T_1 property, then X must have the discrete topology.

   (d) Any infinite subset of ($\mathbb{R}$, cofinite) is dense.

   (e) Let A, B be nonempty **connected** subsets of a subspace X of Euclidean space $\mathbb{R}^n$. If the distance $D(A, B)$ between A and B is zero, then A and B must be contained in the same connected component of X.

(f) Let X be a topological space, and A a dense subset of X. If A is connected, then so is X.

(g) Let X, Y, Z be topological spaces and consider the product $Y \times Z$ with the product topology. Then a function $f : X \to Y \times Z$ is continuous if and only if $f^{-1}(U \times V)$ is open in X for every open subset $U \subseteq Y$ and $V \subseteq Z$.

(h) Any path-connected metric space X is complete.

(i) Let X be a topological space, and $A_1, \ldots, A_n$ a finite collection of subsets of X. If $A_1, \ldots, A_n$ are compact, then so is their union $\bigcup_{i=1}^n A_i$.

(j) A compact subset C of a topological space must contain all of its accumulation points.

(k) A closed and bounded subset of a complete metric space is compact.

2. (5 points) Consider the following statement.

Let X be a topological space, and $A \subseteq X$ a subspace.

If X is ____________, then so is A.

Circle all properties that truthfully fill in the blank. **No justification needed.**

`metrizable` `Hausdorff` T_1 `connected` `disconnected`

`path-connected` `discrete` `indiscrete` `compact` `non-compact`

(By "X is discrete" we mean "X has the discrete topology". Similarly for "indiscrete".)

3. (4 points) For each of the following sequences: state the set of all limits, or, if the sequence has no limits, write "`Does not converge`". **No justification necessary.**

(a) Let $X = \{a, b, c\}$ have the topology $\{\varnothing, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

(i) $a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ \cdots$

(ii) $c,\ c,\ c,\ c,\ c,\ c,\ c,\ c,\ c,\ \cdots$

(b) Let $\mathbb{R}$ have the topology $\{U \mid 0 \notin U\} \cup \{\mathbb{R}\}$.

(i) $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9, \cdots$

(ii) $1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1, \cdots$

4. (24 points) Consider the following topological spaces X and their subsets S. For each set S, circle all terms that apply to S (and its subspace topology). Then compute the interior $\mathrm{Int}(S)$, the closure $\overline{S}$, the boundary ∂S, and the set S' of accumulation points of the set S, viewed as a subset of X. **No justification necessary.**

(a) Let $X = \{a, b, c, d\}$, $\mathcal{T} = \Big\{\varnothing, \{a, b\}, \{c\}, \{a, b, c\}, \{d\}, \{a, b, d\}, \{c, d\}, \{a, b, c, d\}\Big\}$.
Let $S = \{a, c\}$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

$\mathrm{Int}(S)$: __________ $\overline{S}$: __________ ∂S: __________ S': __________

Let $S = \{b\}$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

$\mathrm{Int}(S)$: __________ $\overline{S}$: __________ ∂S: __________ S': __________

(b) Let $X = \mathbb{R}$ and $\mathcal{T} = \Big\{ (a, \infty) \;\Big|\; a \geq 0 \Big\} \cup \{\varnothing, X\}$. **Note the condition $a \geq 0$!**
Let $S = [0, \infty)$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

$\mathrm{Int}(S)$: __________ $\overline{S}$: __________ ∂S: __________ S': __________

Let $S = \{-1\}$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

$\mathrm{Int}(S)$: __________ $\overline{S}$: __________ ∂S: __________ S': __________

(c) Let $X = \mathbb{R}$ and $\{U \mid 0, 1 \notin U\} \cup \{\mathbb{R}\}$. Let $S = \{0, 1\}$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

Int(S): __________ $\overline{S}$: __________ ∂S: __________ S': __________

Let $S = \mathbb{R} \setminus \{0\}$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

Int(S): __________ $\overline{S}$: __________ ∂S: __________ S': __________

5. (3 points) Let $f : X \to Y$ be an **injective** function from a metric space X to a space Y with the cofinite topology. Prove that f is continuous.

6. (3 points) Let X be a topological space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points that converges to a point a_∞ in X. Prove that $A = \{a_n \mid n \in \mathbb{N}\} \cup \{a_\infty\}$ is a compact subset of X.