Final Exam

Math 490
13 December 2024
Jenny Wilson

Name: ______________________________

Instructions: This exam has 6 questions for a total of 50 points.

Each student may bring in one double-sided ($8\frac{1}{2}$" $\times$ 11") sheet of notes, which they must have either hand-written or typed (in font size at least 12) themselves.

The exam is closed-book. No books, additional notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any (non-optional) results proved on the worksheets, on a quiz, or on the homeworks without proof.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	11	
2	5	
3	4	
4	24	
5	3	
6	3	
Total:	50	

1. (11 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let (X, d) be a metric space. Then the function $\tilde{d}(x, y) = \frac{1}{10}d(x, y)$ defines a valid new metric on X.

   True. *Hint:* We can verify directly that, since the three axioms hold for d, they must hold for the function $\frac{1}{10}d$.

   (b) Let X and Y be homeomorphic metric spaces. If X is bounded, then so is Y.

   False. For example, the function $f(x) = \frac{1}{x}$ defines a homeomorphism between the bounded subset $(0, 1)$ of ($\mathbb{R}$, Euclidean) and the unbounded subset $(1, \infty)$ of ($\mathbb{R}$, Euclidean).

   (c) Let X be a **finite** topological space. If X has the T_1 property, then X must have the discrete topology.

   True. *Hint:* We must check that every subset of X is open, equivalently, that every subset of X is closed. In a T_1 space, singleton sets $\{x\} \subseteq X$ are closed (Homework #9 Problem 1(a)). Since X is finite, any subset of X is a finite union of singleton sets. A finite union of closed sets is closed (Worksheet #9 Problem 4(b) and induction).

   (d) Any infinite subset of ($\mathbb{R}$, cofinite) is dense.

   True. *Hint:* The only infinite closed subset of ($\mathbb{R}$, cofinite) is $\mathbb{R}$.

   (e) Let A, B be nonempty **connected** subsets of a subspace X of Euclidean space $\mathbb{R}^n$. If the distance $D(A, B)$ between A and B is zero, then A and B must be contained in the same connected component of X.

   False. Consider $X = \mathbb{R} \setminus \{0\}$. Let $A = (-\infty, 0)$ and $B = (0, \infty)$. Then A and B are two distinct connected components of X, but the distance $D(A, B) = 0$.

(f) Let X be a topological space, and A a dense subset of X. If A is connected, then so is X.

True. *Hint:* Homework #12 Problem 3 shows that if A is connected, then so is $\overline{A}$.

(g) Let X, Y, Z be topological spaces and consider the product $Y \times Z$ with the product topology. Then a function $f : X \to Y \times Z$ is continuous if and only if $f^{-1}(U \times V)$ is open in X for every open subset $U \subseteq Y$ and $V \subseteq Z$.

True. *Hint:* Subsets of the form $U \times V$ form a basis for the product topology, so the result follows from Worksheet #12, Exercise 4.

(h) Any path-connected metric space X is complete.

False. For example, the open interval $(0, 1)$ of the Euclidean real line is path-connected but not complete. The subspace $\mathbb{R}^2 \setminus \{(0, 0)\}$ of Euclidean space $\mathbb{R}^2$ is path-connected, but it is not complete (Worksheet #7 Problem 3).

(i) Let X be a topological space, and $A_1, \dots, A_n$ a finite collection of subsets of X. If $A_1, \dots, A_n$ are compact, then so is their union $\bigcup_{i=1}^n A_i$.

True. *Hint*: For an arbitrary open cover of the union, choose a finite subcover for each A_i. The union of finitely many finite subcovers is finite.

(j) A compact subset C of a topological space must contain all of its accumulation points.

False. Consider $X = \{0, 1\}$ with the indiscrete topology. Then $\{0\}$ is compact (since it is finite) but it does not contain its accumulation point 1.

(k) A closed and bounded subset of a complete metric space is compact.

False. For example, the metric space ($\mathbb{R}$, discrete metric) is complete, since the only Cauchy sequences are the sequences that are (after finitely many terms) constant sequences. It is (viewed as a subspace of itself) closed and bounded, but it is not compact.

2. (5 points) Consider the following statement.

   Let X be a topological space, and $A \subseteq X$ a subspace.

   If X is ____________, then so is A.

   Circle all properties that truthfully fill in the blank. **No justification needed.**

   `metrizable` `Hausdorff` T_1 `connected` `disconnected`

   `path-connected` `discrete` `indiscrete` `compact` `non-compact`

   (By "X is discrete" we mean "X has the discrete topology". Similarly for "indiscrete".)

3. (4 points) For each of the following sequences: state the set of all limits, or, if the sequence has no limits, write "`Does not converge`". **No justification necessary.**

   (a) Let $X = \{a, b, c\}$ have the topology $\{\varnothing, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

   (i) $a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ \cdots$ `Does not converge`

   (ii) $c,\ c,\ c,\ c,\ c,\ c,\ c,\ c,\ c,\ \cdots$ limits: $\{a, b, c\}$

   (b) Let $\mathbb{R}$ have the topology $\{U \mid 0 \notin U\} \cup \{\mathbb{R}\}$.

   (i) $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9, \cdots$ limits: $\{0\}$

   (ii) $1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1, \cdots$ limits: $\{0, 1\}$

4. (24 points) Consider the following topological spaces X and their subsets S. For each set S, circle all terms that apply to S (and its subspace topology). Then compute the interior $\mathrm{Int}(S)$, the closure $\overline{S}$, the boundary ∂S, and the set S' of accumulation points of the set S, viewed as a subset of X. **No justification necessary.**

(a) Let $X = \{a, b, c, d\}$, $\mathcal{T} = \Big\{\varnothing, \{a, b\}, \{c\}, \{a, b, c\}, \{d\}, \{a, b, d\}, \{c, d\}, \{a, b, c, d\}\Big\}$.
Let $S = \{a, c\}$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

$\mathrm{Int}(S)$: $\{c\}$ $\overline{S}$: $\{a, b, c\}$ ∂S: $\{a, b\}$ S': $\{b\}$

Let $S = \{b\}$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

$\mathrm{Int}(S)$: $\varnothing$ $\overline{S}$: $\{a, b\}$ ∂S: $\{a, b\}$ S': $\{a\}$

(b) Let $X = \mathbb{R}$ and $\mathcal{T} = \big\{\, (a, \infty) \,\big|\, a \geq 0\big\} \cup \{\varnothing, X\}$. **Note the condition $a \geq 0$!**
Let $S = [0, \infty)$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

$\mathrm{Int}(S)$: $(0,\infty)$ $\overline{S}$: $\mathbb{R}$ ∂S: $(-\infty,0]$ S': $\mathbb{R}$

Let $S = \{-1\}$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

$\mathrm{Int}(S)$: $\varnothing$ $\overline{S}$: $(-\infty,0]$ ∂S: $(-\infty,0]$ S': $(-\infty,-1)\cup(-1,0]$

(c) Let $X = \mathbb{R}$ and $\{U \mid 0, 1 \notin U\} \cup \{\mathbb{R}\}$. Let $S = \{0, 1\}$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

$\mathrm{Int}(S)$: $\varnothing$ $\overline{S}$: $\{0,1\}$ ∂S: $\{0,1\}$ S': $\{0,1\}$

Let $S = \mathbb{R} \setminus \{0\}$.

`compact` `connected` T_1 T_2 `(Hausdorff)`

$\mathrm{Int}(S)$: $\mathbb{R}\setminus\{0,1\}$ $\overline{S}$: $\mathbb{R}$ ∂S: $\{0,1\}$ S': $\{0,1\}$

5. (3 points) Let $f : X \to Y$ be an **injective** function from a metric space X to a space Y with the cofinite topology. Prove that f is continuous.

Solution. Recall that, in a space Y with the cofinite topology, by definition the open subsets are the sets $\varnothing$ and any set U with a finite complement $Y \setminus U$. Equivalently, the closed sets are precisely the sets Y and any finite set.

To show that f is continuous, by Worksheet #12 Problem 2(a), it suffices to prove that the preimage $f^{-1}(C)$ of any closed set $C \subseteq Y$ is closed in X.

The preimage $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\} = X$ is always closed in X.

If $S = \{y_1, \ldots, y_n\}$ is any finite subset of Y, then $f^{-1}(S) = \bigcup_{i=1}^{n} f^{-1}(\{y_i\})$ is a finite set of at most n elements, by our assumption that f is injective. We proved on Quiz #2 Problem 2 that a singleton sets $\{x\}$ of a metric space X are always closed—it follows that finite subsets of a metric space are closed.

We conclude that the preimage of every closed subset of Y is closed in X. Hence f is continuous.

6. (3 points) Let X be a topological space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points that converges to a point a_∞ in X. Prove that $A = \{a_n \mid n \in \mathbb{N}\} \cup \{a_\infty\}$ is a compact subset of X.

Solution. Let $\{U_i\}_{i\in I}$ be an open cover of A, that is, suppose that U_i is an open subset of X for all $i \in I$, and $A \subseteq \bigcup_{i\in I} U_i$.

Since $\{U_i\}_{i\in I}$ is a cover, there must be some $i_\infty \in I$ such that $a_\infty \in U_{i_\infty}$. Because U_{i_∞} is open, and because the sequence converges to a_∞, by definition, there must exist some $N \in \mathbb{N}$ such that $a_n \in U_{i_\infty}$ for all $n > N$.

Finally, for $n = 1, \dots, N$, there must exist some i_n such that $a_n \in U_{i_n}$. Then, by construction, A is contained in the union $U_{i_\infty} \cup U_{i_1} \cup \cdots \cup U_{i_N}$. We conclude that an arbitrary cover of A has a finite subcover, and therefore that A is compact.