1 Review: Quotient maps and the quotient topology

1.1 The quotient topology

Definition I. (Saturation). If $q : X \to Y$ is a map, and $S \subseteq X$ a subset, the *saturation* of S is the subset $q^{-1}\big(q(S)\big)$ of X. A subset is *saturated* (or *q-saturated*) if it equals its saturation, that is, if it is a union of fibres of q.

Definition II. (Quotient maps and the quotient topology).

- A surjective map $q : X \to Y$ of topological spaces is a *quotient map* if

  $$U \subseteq Y \text{ is open} \qquad \Longleftrightarrow \qquad q^{-1}(U) \text{ is open,}$$

  equivalently, if

  $$C \subseteq Y \text{ is closed} \qquad \Longleftrightarrow \qquad q^{-1}(C) \text{ is closed.}$$

  In other words, q is a surjective, continuous map with the property that the images of saturated open sets are open (equivalently, the images of saturated closed sets are closed).

- Let X be a topological space with equivalence relation $\sim$, and let $X/_\sim$ denote the set of equivalence classes. The *quotient topology* on $X/_\sim$ is the unique topology making the map $q : X \to X/_\sim$ a quotient map. That is, we define a subset $U \subseteq X/_\sim$ to be open if and only if $q^{-1}(U)$ is open in X.

Any quotient map can be viewed as map of the form $q : X \to X/_\sim$ where the equivalence classes are the fibres of q. Observe that the image of a set $A \subseteq X$ is open in the quotient (respectively, closed) if and only if its saturation is open (respectively, closed) in X.

The quotient topology satisfies the following universal property.

Theorem III (Munkres "Topology" Theorem 2.2)**.** *Let $q : X \to X/_\sim$ be a quotient map, and Z any topological space. Let $f : X \to Z$ be a map that is constant on equivalence classes. Then f factors uniquely through a map $f' : X/_\sim \to Z$ to complete the following commutative diagram.*

$$\begin{array}{ccc} X & & \\ \downarrow q & \searrow f & \\ X/_\sim & \overset{f'}{\dashrightarrow} & Z \end{array}$$

The map f' is continuous if and only if f is continuous. The map f' is a quotient map if and only if f is a quotient map.

Exercise 1. Determine which of the following statements about quotient topologies are true or false. If the statement is false, can you modify it to give a true statement?

(i) An injective quotient map is a homeomorphism.

(ii) The composite of quotient maps is a quotient map.

(iii) The restriction of a quotient map to a subspace is a quotient map (onto its image).

(iv) Quotient maps are closed maps.

(v) Quotient maps are open maps.

(vi) Surjective closed maps are quotient maps.

(vii) Surjective open maps are quotient maps.

(viii) Surjective maps from compact spaces to Hausdorff spaces are quotient maps.

(ix) A retraction of X onto a subspace $A \subseteq X$ is a quotient map.

(x) A quotient of a compact space is compact.

(xi) A quotient of a T_1 space is T_1. (Recall T_1 means 'points are closed'.)

(xii) A quotient of a Hausdorff space is Hausdorff.

(xiii) A quotient of a discrete space is discrete.

(xiv) A quotient of a contractible space is contractible.

(xv) Let $f : X \to Y$ be a map and let $\{U_i\}$ be an open cover of X. If $f\big|_{U_i} : U_i \to Y$ is a quotient map onto its image for all i, then so is f.

(xvi) If $f_i : X_i \to Y$ is a quotient map onto its image for all i, then so is $\bigsqcup_i f_i : \bigsqcup_i X_i \to Y$.

(xvii) If $\bigsqcup_i f_i : \bigsqcup_i X_i \to Y$ is a quotient map onto its image, then so is f_i is for all i.

1.2 Homotopies on quotient spaces

Exercise 2. (Homotopies of quotient spaces)

(i) We recall a theorem from Munkres.

Theorem IV (Munkres "Elements of algebraic topology" Theorem 20.1)**.**
Let $p : X \to (X/\sim)$ be a quotient map, and let C be a locally compact Hausdorff space. Then

$$p \times id_C : X \times C \to (X/\sim) \times C$$

is a quotient map.

Deduce that a homotopy $(X/\sim) \times I \to Y$ is continuous if and only if it arises from a continuous map $X \times I \to Y$ which is, for each fixed $t \in I$, constant on equivalence classes in X.

(ii) **(Bonus)** Prove Munkres Theorem 20.1.

(iii) **(Bonus)** Prove that, in general, a product of quotient maps need not be a quotient map. More generally, Theorem IV may fail without the assumption that C is locally compact and Hausdorff. *Hint:* Munkres Ch 22 Ex 7 considers the quotient maps $id : \mathbb{Q} \to \mathbb{Q}$ and $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}_{>0}$ collapsing $\mathbb{Z}_{>0}$ to a point.

1.3 Attaching maps

Definition V. Let X and Y be topological spaces, and let $A \subseteq X$ be a subspace. Given a continuous map $f : A \to Y$, we can *attach X to Y along A via f* to form the space $X \sqcup_f Y$, defined to be the quotient space of $X \sqcup Y$ by the equivalence relation generated by the identifications $a \sim f(a)$ for all $a \in A$.

Exercise 3. Let X and Y be topological spaces, $A \subseteq X$ be a subspace, and $f : A \to Y$ a continuous map. Let $X \sqcup_f Y$ be the space obtained by gluing X to Y along A via f.

(a) Suppose A is closed in X (respectively, open). Prove that the image of Y is closed in $X \sqcup_f Y$ (respectively, open).

(b) Suppose A is either closed or open in X. Show that the map $Y \hookrightarrow X \sqcup Y \to X \sqcup_f Y$ is an embedding (a homeomorphism onto its image). In other words, gluing the space X to Y via f does not induce any relations between points of Y, or alter its topology.

(c) Suppose A is closed or open. Is the map $(X \setminus A) \hookrightarrow X \sqcup Y \to X \sqcup_f Y$ always an embedding?

(d) Suppose f is injective. Is the map $X \to X \sqcup Y \to X \sqcup_f Y$ always an embedding?

(e) **(Bonus)** What happens when A is not closed or open in X? What if f is not continuous?