1 Posets and their Hasse diagrams

Definition I. A *partially ordered set* (often abbreviated to *poset*) $\mathcal{P} = (P, \leq)$ is a set P (called the *ground set*) with a *partial order* $\leq$. A partial order $\leq$ is a *relation* on P (that is, a subset of $P \times P$) that is

- *reflexive*: $a \leq a$ for all $a \in P$.
- *antisymmetric*: if $a \leq b$ and $b \leq a$, then $a = b$.
- *transitive*: if $a \leq b$ and $b \leq c$ then $a \leq c$.

If $a \leq b$ then we say a is *less than* b or a *precedes* b. If $a \leq b$ or $b \leq a$, we say a and b are *comparable*. Otherwise, they are *incomparable*. A poset is *totally ordered* if every pair of elements are comparable.

We write $a < b$ if $a \leq b$ and $a \neq b$.

We say that b *covers* a if $a < b$ and there does not exist any third element x such that $a < x < b$.

A *subposet* of a poset $(P, \leq)$ is a subset of P with the restriction of the partial order $\leq$.

A *chain* in a poset $(P, \leq)$ is a totally ordered subposet $a_0 < a_1 < \cdots < a_p$. The *height* of an element a in a poset is the maximum number p (possibly infinity) such that there exists a chain $a_0 < a_1 < \cdots < a_p = a$ of length $(p+1)$.

Given a poset $(P, \leq)$ and a, b in P, the *interval* $[a, b]$ (also denoted $P_{[a,b]}$) is the subposet

$$P_{[a,b]} = \{x \in P \mid a \leq x \leq b\}.$$

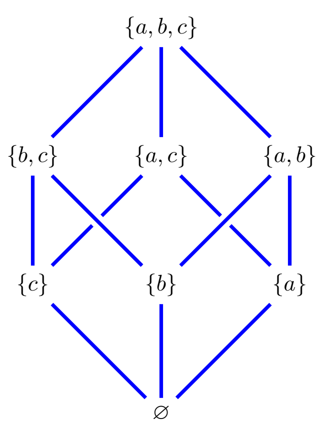
We also define the following notation for the subposets

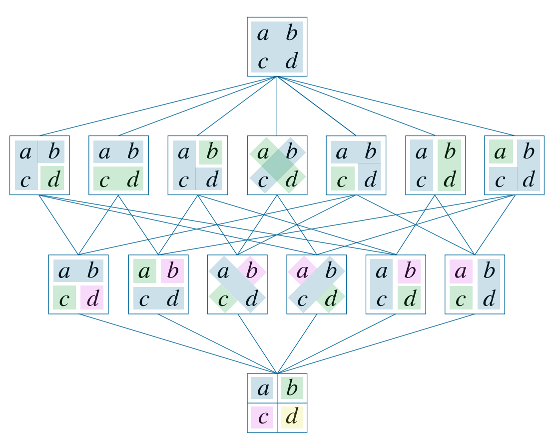
$$P_{<a} = \{x \in P \mid x < a\} \qquad P_{\leq a} = \{x \in P \mid x \leq a\} \qquad P_{\geq a} = \{x \in P \mid x \geq a\} \qquad P_{>a} = \{x \in P \mid x > a\}.$$

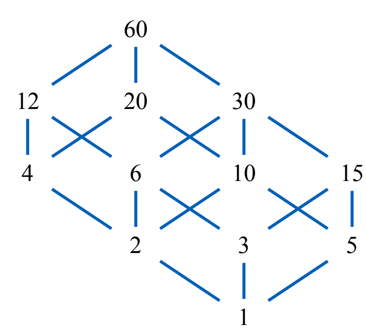
Definition II. A *Hasse diagram* is a directed graph that encodes a finite poset $(P, \leq)$. It consists of a vertex for each element of P, with a directed edge (usually oriented upwards) from $a \in P$ to $b \in P$ whenever b covers a.

It is convenient to write elements of P of height h on row $(h + 1)$ from the bottom of the Hasse diagram.

Example III. The following figures show the Hasse diagram of the poset of subsets of a set $\{a, b, c\}$, ordered by inclusion, and the Hasse diagram of the poset of partitions of the set $\{a, b, c, d\}$, ordered by refinement, and the Hasse diagram of the poset of divisors of 60, ordered by divisibility.







Exercise 1. Consider the set $[3] = \{1, 2, 3\}$ with the usual ordering $1 < 2 < 3$. Draw the Hasse diagrams for the following partial orders on the Cartesian product $[3] \times [3]$.

(a) *Lexicographical order*: $(a, b) \leq (c, d)$ if $a \leq c$, or if $a = c$ and $b \leq d$.

(b) *Product order*: $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$.

(c) *Reflexive closure of strict direct product order*: $(a \leq b) \leq (c, d)$ if $(a, b) = (c, d)$, or if $a < c$ and $b < d$.

1.1 Least elements, minima, and lower bounds

Definition IV. An element a in a poset $(P, \leq)$ is called *minimal* if there is no element x with $x \leq a$. It is called a *least* element if $a \leq b$ for all $b \in P$. *Maximal* elements and *greatest* elements are defined analogously.

A least element, if it exists, is unique. However, posets with no least element may have multiple (incomparable) minimal elements.

Definition V. Let A be a subset of a poset $(P, \leq)$. A *lower bound* of A is an element $\ell \in P$ such that $\ell \leq a$ for all $a \in A$. The element ℓ may or may not be contained in A. A greatest element of the subposet of lower bounds of A is called the *greatest lower bound* of A. *Upper bounds* and *least upper bounds* are defined similarly.

Definition VI. A poset $(P, \leq)$ is called a *lattice* if every pair of elements $\{a, b\} \subseteq P$ has a greatest lower bound (denoted $a \wedge b$) and a least upper bound (denoted $a \vee b$).

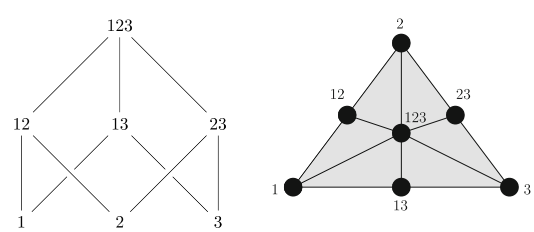
1.2 Maps of posets

Definition VII. Given posets $(P, \leq)$ and $(Q, \leq)$, a function $f : P \to Q$ is called *order-preserving* or *monotone* if $f(a) \leq f(b)$ whenever $a \leq b$. It is called *strictly* order-preserving or *strictly* monotone if $f(a) < f(b)$ whenever $a < b$.

2 The order complex of a poset

Definition VIII. Let $\mathcal{P} = (P, \leq)$ be a poset. The *order complex* $\Delta(\mathcal{P})$ of $\mathcal{P}$ is the abstract simplicial complex whose vertex set is P, and whose simplices are precisely the nonempty finite chains in $\mathcal{P}$.

Example IX. Consider the poset of nonempty subsets of $\{1, 2, 3\}$ ordered by inlcusion. The Hasse diagram and its order complex are illustrated. The geometric realization of the order complex is the barycentric subdivision of a 2-simplex.



Exercise 2. Verify that a monotone map of posets induces a simplicial map on their order complexes.

Definition X. A *flag complex* is an (abstract) simplicial complex X with the following property. Let S be a nonempty subset of vertices of X. If every pair of vertices of S span an edge, then S is a simplex of X.

Exercise 3. Prove that the order complex of a poset is a flag complex.