1 Some families of Δ complexes and (abstract) simplicial complexes

Exercise 1. (Bonus) For the following families of complexes,

(a) Determine the dimensions of each complex. Is the complex pure?

(b) Draw the complexes explicitly in some examples low degrees. Can you determine the homotopy types in these cases?

(c) For the finite complexes: can you determine the Euler characteristic?

(d) Can you describe the link of a simplex?

The following exercise is difficult, and intended to give you an appreciation for the tools we will develop in the second half of the class!

Exercise 2. (Bonus) What can you say about the homotopy type of these complexes? Can you put bounds on their connectivity?

Throughout these notes, for a positive integer n, let $[n] = \{1, 2, 3, \ldots, n\}$.

1.1 n-simplices and their d-skeleta

Below are some familiar examples of simplicial complexes. These (comparatively simple) complexes will serve as our first test cases for applying our various tools for proving simplicial complexes are highly connected.

- The (closed) standard n-simplex Δ^n.
- The d-skeleton of the standard n-simplex Δ^n. In particular its $(n-1)$-skeleton is its boundary $\partial\Delta^n$.
- The barycentric subdivision of Δ^n.
- The barycentric subdivision of the d-skeleton of Δ^n.
- The d-skeleton of the barycentric subdivision of Δ^n.

1.2 Partition complexes

Definition I. Let $n \geq 1$. Let Π_n denote the lattice of partitions of $[n]$, ordered by refinement. We let $\overline{\Pi}_n$ denote the lattice of partitions of $[n]$, excluding the finest partition $[n] = \{1\} \cup \{2\} \cup \ldots \{n\}$ and the coarsest partition $[n] = [n]$. The order complex of $\overline{\Pi}_n$ is called the *partition complex*.

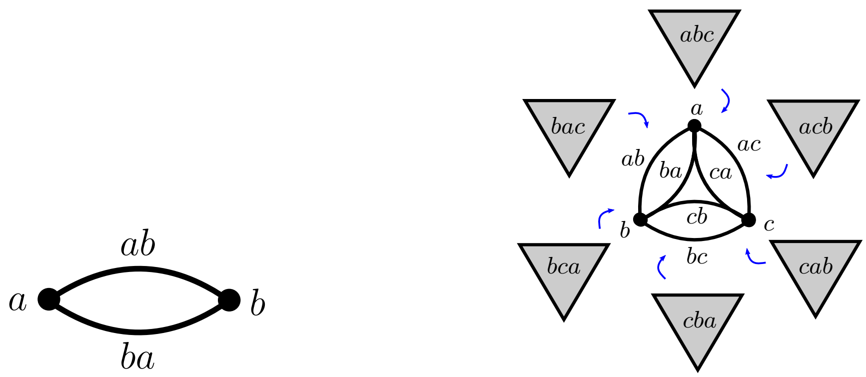
There are a number of variants on the partition complex. We could, for example, consider only partitions into specified numbers of parts. We could consider just *non-crossing* partitions. We could consider partitions of multisets. Another variation:

Definition II. Fix positive integers n and d. The *d-divisible partition lattice* Π_n^d is the sublattice of partitions, all of whose blocks have size divisible by d. We obtain the poset $\overline{\Pi}_n^d$ from Π_n^d by deleting its least and greatest elements (if they exist). The *d-divisible partition complex* is the order complex of $\overline{\Pi}_n^d$.

1.3 The complex of injective words

Let $n \geq 1$. A *word* of length p in $[n]$ is an ordered p-tuple of elements of $[n]$. The components are called *letters* of the word. A word is an *injective word* if its letters are distinct elements of $[n]$. A *subword* of a word w is any word obtained by deleting (not necessarily adjacent) letters of w.

Definition III. The *complex of injective words* I_n is the Δ-complex defined as follows. Its vertices are the letters $1, 2, \ldots, n$. Its p-simplices are injective words of length $(p+1)$. The simplex corresponding to the injective word $a_0 a_1 \cdots a_p$ is glued to the $p+1$ faces $a_0 \ldots \hat{a}_i \ldots a_p$.



(a) Injective words on $\{a, b\}$

(b) Injective words on $\{a, b, c\}$

Figure 1: The complex of injective words

1.4 A complex from number theory

Definition IV. Recall that an integer is *squarefree* if it is a product of distinct primes. For a positive squarefree integer k, let $P(k)$ be the set of its prime factors.

Definition V. Fix an integer $n > 0$. Define an abstract simplicial complex Δ_n by the set of simplices

$$\{P(k) \mid k \text{ is squarefree and } k \leq n\}.$$

Exercise 3. (i) Verify that Δ_n is a well-defined simplicial complex. What are its vertices?

(ii) Justify Björner's [Bj11] description of Δ_n as "the simplicial complex of squarefree positive integers less than or equal to n ordered by divisibility."

Björner [Bj11, Section 1] notes that deep propositions in number theory, including the Prime Number Theorem and the Reimann Hypothesis, are equivalent to statements about the asymptotic rate of growth in n of the Euler characteristic of these complexes.

References

[Bj11] Björner, Anders. "A cell complex in number theory." Advances in Applied Mathematics 46.1-4 (2011): 71-85.

1.5 The Tits buildings

Definition VI. Let V be a vector space over a field k. Consider the poset of proper, nonzero subspaces of V, ordered by inclusion. Its order complex is called the *Tits building* (of type A) and denoted $\mathcal{T}(V)$.

1.6 The complex of partial bases

Definition VII. Let R be a principal ideal domain, and let M be a finite-rank free R-module. A set $\{b_1, \ldots, b_k\}$ in M is called a *partial basis* if it is a subset of a basis for M, equivalently, if it is the basis of a direct summand of M. Let $\mathcal{PB}(M)$ be the poset of partial bases for M under inclusion. The order complex of $\mathcal{PB}(M)$ is called the *partial basis complex* of M.

Consider in particular the case that R is a field and M a finite-dimensional vector space, or the case that $R = \mathbb{Z}$ and M is a finitely generated free abelian group.

1.7 The nerve of an open cover

The nerve of an open cover is a simplicial complex that encodes the combinatorics of the pattern of intersection of the cover.

Definition VIII. Let T be a topological space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of T. The nerve $N(\mathcal{U})$ of the cover is an abstract simplicial complex, defined as follows. Its vertices are the index set I. A finite collection of vertices $\{i_0, i_1, \ldots, i_p\}$ form a p-simplex precisely when the intersection $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}$ is nonempty.

1.8 Simplicial complexes associated to graphs

Let G be an undirected graph.

Definition IX. The *clique complex* $X(G)$ of G has vertices equal to the vertices of G. A set of vertices span a simplex precisely when they form a *clique* in G, that is, every pair of distinct vertices in the set are joined by an edge.

Definition X. The *independence complex* $I(G)$ of G has vertices equal to the vertices of G. A set of vertices span a simplex precisely when they form an *indpendent set* of G, that is, no pair of vertices in the set are joined by an edge.

The independence complex is the clique complex of the complement graph of G.

Definition XI. The *neighbourhood complex* $\mathcal{N}(G)$ of G has vertices equal to the vertices of G. A set of vertices span a simplex precisely when they have a common neighbour.

Definition XII. A *matching* in a graph G is a set edges such that none are loops and no two edges share common vertices. The *matching complex* of G has vertices equal to the edges of G, and simplices equal to the matchings of G.

1.9 New complexes from old

Definition XIII. Let X be an abstract simplicial complex, and S a set (we think of the elements as *labels*). Define the *label complex* X^S to be the following abstract simplicial complex. Its vertices are $V(X) \times S$; we view the vertices of X labelled by elements of S. A set of vertices span a simplex if (after forgetting the labels) the vertices span a simplex of X.

The complex X^S has $|S|^{k+1}$ k-simplices for each k-simplex of X.

Definition XIV. Let X be a simplicial complex. The *ordered complex* $\langle X \rangle$ of X is the Δ-complex defined as follows. The vertex set of $\langle X \rangle$ is the vertex set of X. For each k-simplex of X, we attach $(k+1)!$ many k-simplices to $\langle X \rangle$, one for each ordering of its vertices. The simplex $v_0 v_1 \cdots v_p$ is glued along the $p+1$ faces $v_0 v_1 \dots \hat{v}_i \dots v_p$.

1.10 Additional examples

If there's interest, some additional important families of simplicial complexes that we can study are ...

- Tits buildings of other types
- Coxeter complexes
- Arc complexes (of isotopy classes of arcs on a surface)