1 Joins

Definition I. Let X and Y be topological spaces. The *join* of X and Y, denoted $X \star Y$, is the quotient space of the disjoint union $X \sqcup (X \times Y \times I) \sqcup Y$ obtained by glueing $(X \times Y \times I)$ to X along $X \times Y \times \{0\}$ via the projection map p_X, and glueing $(X \times Y \times I)$ to Y along $X \times Y \times \{1\}$ via the projection map p_y,

$$p_X : X \times Y \times \{0\} \longrightarrow X \qquad p_Y : X \times Y \times \{1\} \longrightarrow Y$$
$$(x, y, 0) \longmapsto x \qquad (x, y, 1) \longmapsto y$$

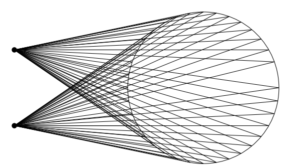
By convention, the join $X \star \varnothing$ of a space X with the empty set is X.

We can think of the join of X and Y as a space constructed from the disjoint union of X and Y by attaching a line segment from every point in X to every point in Y. Specifically, $x \in X$ and $y \in Y$ are joined by the line $\{(x, y, t) \mid 0 \leq t \leq 1\}$. For subspaces of $\mathbb{R}^n$, we can make this construction precise with the following definition.

Proposition II. *Let X and Y be disjoint subsets of some Euclidean space $\mathbb{R}^n$. Then*

$$X \star Y \cong \{t \cdot x + (1 - t) \cdot y \mid x \in X,\ y \in Y, t \in [0, 1]\}.$$

Example III. The join of S^0 and S^1 is homeomorphic to S^2.



Proposition IV. *The join operation satisfies the following properties.*

- *Commutativity: there is a natural isomorphism $X \star Y \cong Y \star X$ for all topological spaces X, Y.*
- *Associativity: there is a natural isomorphism $X \star (Y \star Z) \cong (X \star Y) \star Z$ for all topological spaces X, Y, Z.*

Proposition V. *Let X be a topological space. Let S^n denote the n-sphere, and Δ^n denote an n-simplex.*

(a) The join $X \star \Delta^0$ of X and a point is homeomorphic to the cone CX on X.

(b) The join of X with the 0-sphere S^0 is homeomorphic to the (unreduced) suspension SX of X.

(c) $S^n \star S^m \cong S^{n+m+1}$.

(d) The $(n+1)$-fold join of a 0-sphere $(S^0)^{\star(n+1)} = S^0 \star S^0 \star \cdots \star S^0$ is homeomorphic to S^n.

(e) $\Delta^n \star \Delta^m \cong \Delta^{m+n+1}$.

(f) The $(n+1)$-fold join of a point $(\Delta^0)^{\star(n+1)} = \Delta^0 \star \Delta^0 \star \cdots \star \Delta^0$ is homeomorphic to Δ^n.

Exercise 1. (Bonus) Explain the sense in which the join contains canonical homeomorphic copies of X and Y.
Hint: Worksheet #1, Exercise 3 part (b).

Exercise 2. (Bonus) Prove Proposition IV.

Exercise 3. (Bonus) Verify that the join of CW complexes X and Y has a CW complex structure, and contains X and Y as subcomplexes. Assuming X and Y are CW complexes with the property that the product and weak topology on $X \times Y$ agree, verify that the weak topology agrees with the defining quotient topology on the join.

Exercise 4. (Bonus) Prove Proposition II.

Exercise 5. (Bonus) Prove Proposition V.

1.1 Homotopy type of a join

Proposition VI. *Suppose that there are homotopy equivalences $X \simeq X'$ and $Y \simeq Y'$. Then there is a homotopy equivalence $(X \star Y) \simeq (X' \star Y')$.*

Corollary VII. *The join of any space X with a contractible space is contractible.*

Exercise 6. (Bonus) Prove Proposition VI. *Hint:* Worksheet #1, Exercise 2.

1.2 The homology and homotopy groups of a join

Proposition VIII. *Let X and Y be topological spaces. Then*

$$\widetilde{H}_{k+1}(X \star Y) \cong \bigoplus_{i+j=k} \widetilde{H}_i(X) \otimes \widetilde{H}_i(Y) \oplus \bigoplus_{i+j=k-1} \mathrm{Tor}(\widetilde{H}_i(X), \widetilde{H}_j(Y)).$$

Proposition IX. *For $i = 1, \ldots, k$ let X_i be an n_i-connected topological space. Then the join $X_1 * X_2 * \cdots * X_p$ is $((\sum_{i=1}^{p}(n_i + 2)) - 2)$-connected.*

Exercise 7. (Bonus) In this exercise we will prove Proposition VIII, using the Künneth formula.

Theorem X (The Künneth formua)**.** . *Given a PID R and topological spaces X and Y the homology of their product is determined by the following (noncanonically) split short exact sequence of R-modules.*

$$0 \longrightarrow \bigoplus_{i+j=k} H_i(X;R) \otimes_R H_j(Y;R) \xrightarrow{\varphi} H_k(X \times Y;R) \longrightarrow \bigoplus_{i+j=k-1} \mathrm{Tor}_1^R(H_i(X;R), H_j(Y;R)) \longrightarrow 0$$

(a) Let A be the image $X \sqcup X \times Y \times \left[0, \frac{3}{4}\right)$ in $X \star Y$. Let B be the image of $Y \sqcup X \times Y \times \left(\frac{1}{4}, 1\right]$ in $X \star Y$. Verify that A and B are open subsets of $X \star Y$.

(b) Verify that $A \cup B = X \star Y$, A deformation retracts onto (a homeomorphic copy of) X, B deformation retracts onto (a homeomorphic copy of) Y, and $A \cap B$ deformation retracts onto (a homeomorphic copy of) $X \times Y$.

(c) Write the Mayer–Vietoris sequence on homology associated to the open cover A, B of $X \star Y$.

(d) Verify that the inclusion maps $X \hookrightarrow X \star Y$ and $Y \hookrightarrow X \star Y$ are nullhomotopic.

(e) Deduce that the Mayer–Vietoris sequence simplifies to the short exact sequences

$$0 \longrightarrow H_{k+1}(X \star Y) \longrightarrow H_k(X \times Y) \xrightarrow{\psi} H_k(X) \oplus H_k(Y) \longrightarrow 0$$

(f) Conclude the theorem.

Exercise 8. In this exercise we will prove Proposition IX, following Section 2 of Milnor "Construction of Universal Bundles, II".

(a) Verify that, by induction, it suffices to prove the result in the case $p = 2$. Thus we suppose X is an n_X-connected space and Y is an n_Y-connected space. We wish to prove that $X \star Y$ is $(n_X + n_Y + 2)$-connected.

(b) Verify the result in the case that one of X or Y is empty. This establishes the case that n_X or n_Y is (-2).

(c) Show that if X and Y are nonempty, then $X \star Y$ is path-connected. This establishes the case that $n_X, n_Y = -1$.

(d) **(Bonus)** Suppose that X is nonempty and Y is path-connected. Show $X \star Y$ is simply connected. This establishes the case that $\{n_X, n_Y\} = \{-1, 0\}$.

(e) To finish the proof, verify the result in the case that n_X and n_Y are both nonnegative. *Hint: $X \star Y$ is simply-connected by the previous part.* Apply Proposition VIII and a suitable version of Whitehead's theorem.

1.3 Joins of simplicial complexes

Definition XI. Let X and Y be abstract simplicial complexes with vertex sets $V(X)$ and $V(Y)$, and simplices $S(X)$ and $S(Y)$. The *join* of X and Y is the abstract simplicial complex $X \star Y$ with vertex set and simplices

$$V(X \star Y) = V(X) \cup V(Y) \qquad S(X \star Y) = S(X) \cup S(Y) \cup \{\sigma \cup \tau \mid \sigma \in S(X),\ \tau \in S(Y)\}.$$

The following proposition states that this construction does indeed correspond to the join of topological spaces.

Proposition XII. *Let X and Y be abstract simplicial complexes. Then*

$$|X \star Y| \cong |X| \star |Y|.$$

Exercise 9. (Bonus) For abstract simplicial complexes X and Y, verify that the join $X \star Y$ satisfies the definition of an abstract simplicial complex.

Exercise 10. (Bonus) Verify Proposition XII.

Exercise 11. Let $\mathcal{P} = (P, \leq)$ be a poset, and $\Delta(\mathcal{P})$ be its order complex.

(a) For $x \in P$, prove that the link of the vertex x in $\Delta(\mathcal{P})$ is the join $\Delta(\mathcal{P}_{<x}) \star \Delta(\mathcal{P}_{>x})$.

(b) Let σ be a p-simplex in $\Delta(\mathcal{P})$ corresponding to the chain $x_0 < x_1 < \cdots < x_p$. Describe the link of σ as an iterated join.

Exercise 12. Let X be a simplicial complex and σ a simplex. Prove or disprove: $\mathrm{Star}_X(\sigma) = \sigma \star \mathrm{Lk}_X(\sigma)$.