# 1 Subdivisions of simplicial complexes

**Definition I.** Let $X$ be a $\Delta$-complex. A *subdivision* of $X$ is a new $\Delta$-complex structure $X'$ on the underlying topological space of $X$ such that each simplex of $X'$ is contained in a simplex of $X$, and every simplex of $X$ is a finite union of simplices of $X'$.

By this definition, the underlying spaces of $X$ and $X'$ agree as sets and as topological spaces.

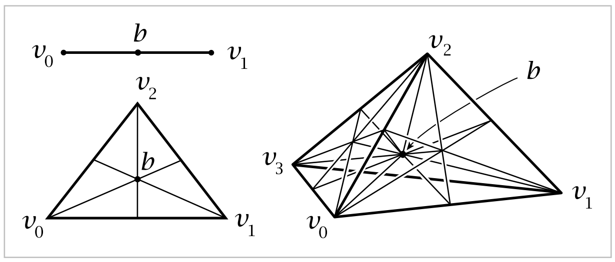
## 1.1 Barycentric subdivision

The barycentric subdivision of a complex is obtained by introducing a new vertex at the barycentre (i.e. centre of mass) of each simplex and taking the coarsest associated subdivision. Each $n$-simplex is replaced by a union of $(n+1)!$ many $n$-simplices.

**Definition II.** Given an $n$-simplex $\Delta^n_\alpha$ with vertices $v_0, v_1, \dots, v_n$, the *barycentre* of $\Delta^n_\alpha$ is the point

$$\left(\frac{1}{n+1}\right)v_0 + \left(\frac{1}{n+1}\right)v_1 + \cdots + \left(\frac{1}{n+1}\right)v_p.$$

We define the barycentric subdivision of a simplex inductively. The subdivision of a 0-simplex $\{v_0\}$ is simply the vertex $\{v_0\}$. The barycentric subdivision of the $n$-simplex $\{v_0, v_1, \dots, v_n\}$ is constructed by placing a new vertex $b$ at the barycentre and subdividing into a union of $n$-simplices of the form $\{b, w_0, \dots, w_{n-1}\}$, where $\{w_0, \dots, w_{n-1}\}$ is an $(n-1)$-simplex in the barycentric subdivision of a face $\{v_0, \dots, \hat{v}_i, \dots, v_n\}$ of the simplex. The illustration (from Hatcher) shows the cases $n = 1, 2, 3$.




If $X$ is a $\Delta$-complex, its *barycentric subdivision*—which we will denote $\mathrm{sd}(X)$—is a $\Delta$-complex constructed by inductively gluing in, in place of each $n$-simplex of $X$, a barycentrically subdivided $n$-simplex.

**Exercise 1. (Bonus)** Verify the details of Definition II: check that the construction of $\mathrm{sd}(X)$ yields a valid $\Delta$-complex structure, that is homeomorphic to $X$, and that it is a subdivision of $X$ in the sense of Definition I.

**Exercise 2. (Bonus)** Let $X$ be a $\Delta$-complex. Prove or disprove: The barycentric subdivision of the $k$-skeleton of $X$ is the $k$-skeleton of $\mathrm{sd}(X)$.

**Proposition III.** *Let $X$ be a simplicial complex. Let $\mathcal{S} = (S, \subseteq)$ be the set of simplices of $X$, ordered by inclusion. Then the realization of the order complex $|\mathcal{S}|$ is canonically isomorphic to the barycentric subdivision of $X$.*

In light of Proposition III, we can define the barycentric subdivision of an abstract simplicial complex as the order complex on its poset of simplices under inclusion.

**Definition IV.** Let $X$ be an abstract simplicial complex. The *barycentric subdivision* $\mathrm{sd}(X)$ of $X$ is an abstract simplicial complex defined as follows. The vertex set of $\mathrm{sd}(X)$ is the set of simplices $S(X)$ of $X$. A collection of simplices $\{\sigma_0, \sigma_1, \dots, \sigma_p\}$ of $X$ form a $p$-simplex of $\mathrm{sd}(X)$ precisely when they form a chain under inclusion.

**Exercise 3.** Prove Proposition III.

**Exercise 4. (Bonus)** Let $f : X \to Y$ be a simplicial map of abstract simplicial complexes. Show that $f$ induces a simplicial map $f_* : \mathrm{sd}(X) \to \mathrm{sd}(Y)$ on the barycentric subdivisions of $X$ and $Y$. Prove moreover that barycentric subdivision defines an endofunctor on the category of abstract simplicial complexes and simplicial maps.

**Exercise 5.** Verify that Definitions II and IV are compatible: given an abstract simplicial complex $X$, the realization of its barycentric subdivision $|\mathrm{sd}(X)|$ is the (geometric) barycentric subdivision of its realization $\mathrm{sd}(|X|)$.

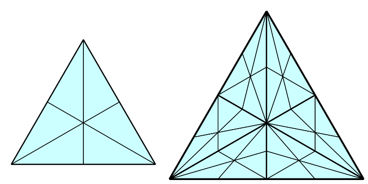
**Exercise 6.** Let $\mathcal{P}$ be the poset of nonempty subsets of $[n+1]$, ordered by inclusion. Show that its geometric realization $|\mathcal{P}|$ is the barycentric subdivision of an $n$-simplex. Verify that it has $(n+1)!$ top-dimensional simplices.

The following proposition shows that a $\Delta$-complex structure can be refined to a simplicial complex structure by repeated barycentric subdivision.

**Proposition V.**
- *Let $X$ be a $\Delta$-complex. Then its barycentric subdivision $\mathrm{sd}(X)$ has the property that every $n$-simplex has $(n+1)$ distinct vertices.*
- *Let $X$ be a $\Delta$-complex with the property that every $n$-simplex has $(n+1)$ distinct vertices. Then its barycentric subdivison $\mathrm{sd}(X)$ is a simplicial complex.*

*Consequently, the two-fold barycentric subdivision $\mathrm{sd}^2(X)$ of any $\Delta$-complex $X$ is a simplicial complex.*

The following figure (from Wikipedia) shows the first and second subdivision of a 2-simplex.



**Exercise 7.** (a) Prove Proposition V.

(b) Show by example that a single barycentric subdivision of a $\Delta$-complex may not be a simplicial complex.

**Exercise 8. (Bonus)** Describe the barycentric subdivision of the complex of injective words (see Worksheet #11, Definition III). Verify that it is a simplicial complex.

**Exercise 9. (Bonus)** Formulate a definition of a barycentric subdivision of a poset $(P, \leq)$, so that the order complex of the barycentric subdivision is the barycentric subdivision of the order complex of $(P, \leq)$. *Hint*: See Kozlov Section 10.3.5.

**Exercise 10. (Bonus)** Let $X$ be a simplicial complex and $\mathrm{sd}(X)$ its barycentric subdivision. Discuss: can you characterize links of simplices in $\mathrm{sd}(X)$ in terms of the combinatorics of $X$?