1 Cones and near-cones

1.1 Cone points

Definition I. Let X be an abstract simplicial complex. A vertex v_0 of X is a *cone point* or *apex* of X if, for every simplex σ of X, the set $\{v_0\} \cup \sigma$ is a simplex of X. If X has a cone point, we call it a *cone*.

Equivalently, v_0 is a cone point of X if $X = \mathrm{Star}_X(v_0)$. This implies (by Worksheet #12 Exercise 11)

$$X = v_0 \star \mathrm{Lk}_X(v_0) = \mathrm{Cone}(\mathrm{Lk}_X(v_0)).$$

Notably, a complex with a cone point is contractible.

Exercise 1. **(Bonus)** Suppose that X has a cone point v_0. Construct an explicit deformation retraction from X to v_0, and verify that it is continuous.

Exercise 2. **(Bonus)** Show that every vertex of a simplex Δ^n is a cone point. In particular, a complex may have more than one distinct cone points.

Exercise 3. **(Bonus)** Let V be an infinite set, and let X be the 'infinite simplex' on V, i.e., the abstract simplicial complex for which all finite nonempty subsets $\sigma \subseteq V$ are simplices. Verify that X is contractible.

Exercise 4. Let X be a Δ-complex, and consider the analogous definition: Let v_0 be a vertex of X with the property that, for any simplex σ of X, either v_0 is vertex of σ, or $v_0 \star \sigma$ is a simplex of X. Show by example that X is not necessarily contractible.

1.2 Near-cones

The definition of a near-cone, and description of its homotopy type, are (mildly modified) from Björner–Kalai "An extended Euler–Poincaré theorem".

Definition II. Let X be an abstract simplicial complex with vertex set $V(X)$, and let $v_0 \in V(X)$ be a distinguished vertex. The complex X is a *near-cone* if for every simplex σ, if $v_0 \notin \sigma$, then for any vertex $w \in \sigma$ the set $(\sigma \setminus \{w\}) \cup \{v_0\}$ is a simplex of X. For a near-cone X define the notation

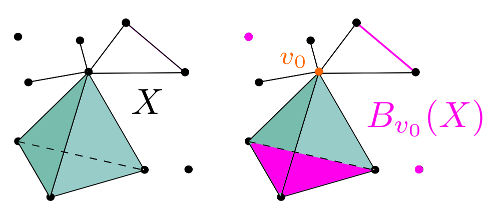
$$B_{v_0}(X) = \{\sigma \in S(X) \mid \sigma \cup \{v_0\} \notin S(X)\}$$

for the set of simplices of X not contained in $\mathrm{Star}_X(v_0)$.

In other words, X is constructed by gluing each simplex in $B_{v_0}(X)$ to $\mathrm{Star}_X(v_0)$ along its (entire) boundary. Since

$$X \simeq X/\mathrm{Star}_X(v_0)$$

by Worksheet #3 Corollary VIII, we can deduce that X is homotopy equivalent to a wedge of spheres (of potentially different dimensions). A 2-dimensional near-cone X is shown.



Theorem III. *Let X be a simplicial complex. If X is a near-cone with v_0 and $B_{v_0}(X)$ as in Definition II, then*

$$X \simeq \bigvee_{\substack{\sigma \in B_{v_0}(X) \\ dim(\sigma)=p}} S^p$$

In particular, the Betti numbers of X are determined by the formula

$$\textit{rank}\left(\widetilde{H}_p(X)\right) = \#\{\sigma \in B_{v_0}(X) \mid \textit{dim}(\sigma) = p\}.$$

Exercise 5. Let X be a near-cone. Show that every simplex of $B_{v_0}(X)$ is a maximal simplex (under inclusion) of X.

Exercise 6. (a) Prove Theorem III. *Hint*: First verify $X = B^{v_0}(X) \cup (\{v_0\} * C)$ for a suitably-defined subcomplex C.

(b) Write down a formula for the Euler characteristics of X.

By Worksheet #7 Exercise 5 we know that the k-skeleton of an n-simplex is either contractible or homotopy equivalent to a wedge of k-spheres. The concept of a near-cone gives a convenient framework to compute its homotopy type: to reprove this fact, and determine the number of spheres.

Exercise 7. Let $n \geq 0$ and $0 \leq k \leq n$. Let $\Delta^{n,k} := (\Delta^n)^{(k)}$ denote the k-skeleton of an n-simplex.

(a) Determine the homotopy type of $\Delta^{n,k}$.

(b) Describe a basis for $H_k\left(\Delta^{n,k}\right)$ of simplicial k-cycles.

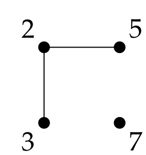
(c) **(Bonus)** The action of the symmetric group on the vertex set $[n+1]$ of Δ^n extends to a simplicial action on its k-skeleton $\Delta^{n,k}$. What can you say about the homology group $H_k\left(\Delta^{n,k}\right)$—or the rational homology group $H_k\left(\Delta^{n,k};\mathbb{Q}\right)$—as an S_{n+1}-representation?

Exercise 8. **(Bonus)** Let $n \geq 0$ and $0 \leq k \leq n$.

(a) Let X be a k-skeleton $(\mathrm{sd}(\Delta^n))^{(k)}$ of the barycentric subdivision $\mathrm{sd}(\Delta^n)$ of Δ^n. Is X a near-cone?

(b) Let X be the barycentric subdivision $\mathrm{sd}\left(\Delta^{n,k}\right)$ of the k-skeleton $\Delta^{n,k}$ of Δ^n. Is X a near-cone?

Björner gives another application of this concept in "A cell complex in number theory" (Theorem 3.1). Fix $n \geq 0$, and recall from Worksheet #11 Section 1.4 that Δ_n is the abstract simplicial complex with vertices the set of positive primes $p \leq n$, and with simplices $\{p_0, p_1, \ldots, p_k \mid p_0 p_1 \cdots p_k \leq n\}$. We can identify $\mathrm{sd}(\Delta_n)$ with the order complex of the poset of squarefree integers $\leq n$ ordered by divisibility. The complex Δ_{10} is shown.



Björner relates the Euler characteristics of the complexes Δ_n to deep results and conjectures in number theory, including the Prime Number Theorem and the Riemann Hypothesis.

Exercise 9. **(Bonus)** Draw Δ_{15}, Δ_{20}, and Δ_{30}.

Exercise 10. Give a description of the homotopy type and the Euler characteristic of Δ_n.