1 Shellable Complexes

Definition I. Following Kozlov, we call a Δ-complex X a *generalized simplicial complex* if for all $n \geq 1$ the attaching map $\phi^n_\alpha : \partial\Delta^n_\alpha \to X$ of any n-simplex Δ^n_α is an embedding of the boundary of Δ^n_α into X. Such a complex is *pure of dimension* d if its maximal simplices (under inclusion) are all d-dimensional.

Simplicial complexes are examples of generalized simplicial complexes, as are families like the complexes of injective words (Worksheet #11 Section 1.3).

Exercise 1. (Bonus) Prove that a Δ-complex X is a generalized simplicial complex if and only if every n-simplex of X has $(n+1)$ distinct vertices, for all n.

Exercise 2. Let Δ^n denote an n-simplex. Let A be a subcomplex of $\partial\Delta^n$ that is pure of dimension $(n-1)$.

(a) Verify that A is the union of a nonempty subset of the $(n+1)$ many $(n-1)$-faces of Δ^n.

(b) Suppose A is a proper subset of $\partial\Delta^n$. Show that A is contractible. *Hint:* Find a cone point.

Exercise 3. Let X be a generalized simplicial complex with $\pi_{n-1}(X) \cong 0$. Let Δ^n be an n-simplex, A a subcomplex of $\partial\Delta^n$ that is pure of dimension $(n-1)$, and $\phi : A \to X$ a simplicial embedding. Consider the space $X \sqcup_\phi \Delta^n$ obtained by gluing Δ^n to X along A. Verify that $X \sqcup_\phi \Delta^n$ is a generalized simplicial complex, and

$$X \sqcup_\phi \Delta^n \simeq \begin{cases} X \vee S^n, & A = \partial\Delta^n \\ X, & A \neq \partial\Delta^n \end{cases}$$

Hint: Worksheet #3 Corollaries X and XI.

Definition II. A finite generalized simplicial complex X is called *shellable* if its maximal simplices can be arranged in linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex $\left(\bigcup_{i=1}^{k-1} F_i\right) \cap F_k$ is pure of dimension $(\dim(F_k) - 1)$ for all $k = 2, \ldots, t$. Such an order on the maximal simplices is called a *shelling order*. For a given shelling order, a maximal simplex F_k is called a *spanning* simplex if $\left(\bigcup_{i=1}^{k-1} F_i\right) \cap F_k$ is the entire boundary of F_k.

Theorem III. *Let X be a generalized simplicial complex. If X is shellable, then*

$$X \simeq \bigvee_{F \text{ spanning}} S^{\dim(F)}.$$

Notably, if the shelling order on X has no spanning simplices, then X is contractible.

Exercise 4. Let X be a shellable generalized simplicial complex and $F_1, \ldots, F_t$ a shelling order on its maximal simplices. Show that, if we re-order the simplices F_k so that (i) the spanning simplices occur after all the non-spanning simplices, and (ii) the spanning simplices occur in any order, the result is another shelling order. Deduce that we may assume (without loss of generality) that the shelling order is such that the spanning simplices occur last and are non-increasing in dimension.

Exercise 5. Use Exercises 4 and 3 to prove Theorem III.

Corollary IV. *Let X be a generalized simplicial complex that is pure of dimension d. If X is shellable, then*

$$X \simeq \bigvee_{\text{spanning faces}} S^d$$

In particular, X is $(d-1)$-connected. The complex X is homotopy equivalent to a wedge of d-spheres (if it has at least one spanning simplex) or contractible (if it has none).

Exercise 6. Let $\Delta^{n,k} = (\Delta^n)^{(k)}$ be the k-skeleton of an n-simplex. Find a shelling order on the maximal simplices of $\Delta^{n,k}$, and use Theorem III to give a new proof of its homotopy type.

Recall from Worksheet #11 Definition III we defined the *complex of injective words* I_n as the following Δ-complex. Its vertex set is $[n] := \{1, 2, \ldots, n\}$. Each p-simplex corresponds to an *injective word* of length $(p+1)$ on $[n]$, that is, an ordered tuple of $(p+1)$ distinct elements of $[n]$. The $(p+1)$ codimension-1 faces of the simplex labelled by the word $a_0 a_1 a_2 \cdots a_p$ are glued to the $(p+1)$ simplices $a_0 \ldots a_{i-1} \hat{a}_i a_{i+1} \ldots a_p$.

Theorem V. *The complex of injective words I_n is a pure generalized simplicial complex of dimension $(n-1)$, and is shellable. There is a homotopy equivalence to a wedge of $(n-1)$-spheres*

$$I_n \simeq \bigvee S^{n-1}$$

with one sphere for each derangement of $[n]$.

Exercise 7. In this exercise we will prove Theorem V.

(a) Verify that the complex of injective words I_n is a generalized simplicial complex.

(b) Verify that its maximal simplices are $(n-1)$-dimensional and correspond to injective words of length n, i.e., to permutations of $[n]$.

(c) The order $1 < 2 < \cdots < n$ on $[n]$ induces the lexicographical (i.e. dictionary) order on the set of injective words. Show that the lexicographical order on the injective words of length n is a shelling order on the maximal simplices of I_n. Conclude that I_n is $(n-2)$-connected.

(d) Find a combinatorial condition that determines whether an injective word corresponds to a spanning simplex.

(e) Show that a basis for $\widetilde{H}_{n-1}(I_n)$ is in bijection with the set of permutations on $[n]$ with no fixed points (called the set of *derangements* of $[n]$). Unfortunately this bijection is not equivariant with respect to the natural simplicial action of the symmetric group on I_n.

Exercise 8. (Bonus) The k-skeleton of I_n is the complex on injective words of $[n]$ of length at most $(k+1)$. Theorem V and Worksheet #7 Exercise 5 imply that the k-skeleton is $(k-1)$-connected. Prove this result directly using shellability. Can you compute the Betti numbers of $I_n^{(k)}$?

Exercise 9. (Bonus) Formulate a more general notion of shellability—that applies to infinite simplicial complexes—wherein we can glue on a set of maximal simplices at each step $1, 2, \ldots, t$.
Remark: We can use such a variation to prove that spherical buildings, such as the classical Tits buildings, are homotopy equivalent to a wedge of spheres.