1 PL Morse Theory

This worksheet is based (in part) on Bestvina's notes "PL Morse Theory". We will later prove the Theorem I as a special case of the "badness" technique (following Hatcher–Vogtmann) covered on upcoming Worksheet #18.

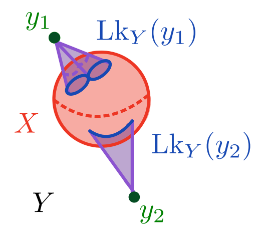
Theorem I. *Let Y be a simplicial complex, and $X \subseteq Y$ a subcomplex. Assume X and Y satisfy the following.*

(i) X is a full subcomplex of Y.

(ii) X is d-connected.

(iii) For all vertices y_1, y_2 in $Y \setminus X$, there is no edge $\{y_1, y_2\}$.

(iv) For all vertices $y \in Y \setminus X$, the link $\mathrm{Lk}_Y(y)$ is $(d-1)$-connected.



Then Y is d-connected.

Under the assumptions of the theorem, the complex Y is constructed from X by attaching a cone on $\mathrm{Lk}_Y(y) = \mathrm{Lk}_Y(y) \cap X$, with apex the vertex y, for each vertex $y \in Y \setminus X$.

Exercise 1. (Bonus) Prove Theorem I directly.

Exercise 2. (Bonus) Let $0 \leq k \leq n$ and let $\Delta^{n,k} = (\Delta^n)^{(k)}$. Consider the subcomplexes $\Delta^{n-1,k} \subseteq \Delta^{n,k}$ and $\mathrm{Lk}_{\Delta^{n,k}}(n)$. Use Theorem I and induction on n to conclude that $\Delta^{n,k}$ is $(k-1)$-connected.

Exercise 3. Prove: in a flag complex (such as the order complex of a poset), the star of a vertex is a full subcomplex.

Exercise 4. Let $0 \leq k \leq n$ be integers. Let $X^{n,k} = \mathrm{sd}\left((\Delta^n)^{(k)}\right)$ be the barycentric subdivision of the k-skeleton of an n-simplex. This is the order complex of the poset of subsets of $[n+1]$ of cardinality at most $(k+1)$, ordered by inclusion. As a warm-up application of Theorem I, we will re-prove that $X^{n,k}$ is $(k-1)$-connected. We proceed by induction on n, and induction on a filtration of $X^{n,k}$. Observe that Δ^0 is a point, hence is d-connected for all d. Fix $n \geq 1$ and assume by induction that the result holds for all $n' < n$ and all $0 \leq k \leq n'$.

(a) We define a filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{k+1} = X^{n,k}$ as follows. At each step, verify the description of the newly added vertices, and vertify that there are no edges between the newly added vertices.

- $X_0 = \mathrm{Star}_{X^{n,k}}(1)$
- X_1 is the full subcomplex on X_0 and all vertices corresponding to subsets of $[n+1]$ of cardinality 1 not contained in X_0, i.e, all vertices $\{a\}$ with $a \neq 1$.
- X_2 is the full subcomplex on X_1 and all subsets of cardinality 2 not contained in X_1, i.e., all vertices $\{a_1, a_2\}$ with $1 \neq a_1, a_2$.

  $\vdots$
- X_i is the full subcomplex on X_{i-1} and all subsets of $[n+1]$ of cardinality i not contained in X_{i-1}, i.e., vertices $\{a_1, \ldots, a_i\} \not\ni 1$.

  $\vdots$
- $X_{k+1} = X^{n,k}$

(b) For $i = 1, 2, \ldots k$, a vertex V in $X_i \setminus X_{i-1}$ is a subset of $[n+1]$ of cardinality i not containing the letter 1. Describe $\mathrm{Lk}_{X_i}(V)$, and verify that the vertex $V \cup \{1\}$ is a cone point of the link. (Why must we assume $i \leq k$?)

(c) The subspace X_0 is a cone (with vertex 1) and thus is contractible. Apply Theorem I inductively to deduce that $X_1, X_2, \ldots$, and X_k are contractible.

(d) Verify that, if $n = k$, then $X_k = X^{n,k}$. Conclude the result in the case $n = k$.

(e) Suppose $n > k$. Let V be a vertex in $X_{k+1} \setminus X_k$. Show $\mathrm{Lk}_{X_{k+1}}(V) \cong X^{k,k-1}$.

(f) Use Theorem I and the inductive hypothesis on n to conclude Theorem II.

Our main application is a proof of the Solomon–Tits theorem in type A, a central result in the study of arithmetic groups. Let V be a finite-dimensional vector space over a field k. Recall from Worksheet #11 Section 1.5 that the *Tits building* (of type A), denoted $\mathcal{T}(V)$, is the order complex of the poset of proper, nonzero subspaces of V, ordered by inclusion.

Theorem II (Solomon–Tits Theorem, type A)**.** *Let V be an n-dimensional vector space. The Tits building $\mathcal{T}(V)$ is $(n-3)$-connected.*

Since $\mathcal{T}(V)$ is not contractible, by Worksheet #7 Corollary V, Theorem II is equivalent to the statement that $\mathcal{T}(V)$ is a wedge of $(n-2)$-spheres.

Exercise 5. In this exercise we will prove Theorem II. The proof follows Section 5.1 of Bestvina's note. We proceed by induction on $\dim(V)$, and induction on a filtration of $\mathcal{T}(V)$.

(a) Verify that Theorem II holds when $\dim(V) = 1$.

(b) Let V be a k-vector space of dimension $n \geq 2$, and assume by induction the theorem holds for vector spaces of dimension $1, 2, \ldots, n-1$. Fix a line $L_0 \subseteq V$, and define a filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{n-1} = \mathcal{T}(V)$ as follows. Verify, at each step, the description of the newly added vertices, and vertify that there are no edges between the newly added vertices.

- $X_0 = \mathrm{Star}_{\mathcal{T}(V)}(L_0)$
- X_1 is the full subcomplex on X_0 and all lines $L \subsetneq V$ not in X_0. These are all lines satisfying $L \neq L_0$.
- X_2 is the full subcomplex on X_1 and all planes $P \subsetneq V$ not in X_0. These are all planes satisfying $P \not\supseteq L_0$.

  $\vdots$
- X_i is the full subcomplex on X_{i-1} and all subspaces $W \subsetneq V$ of dimension i not contained in X_0, that is, all i-dimensional subspaces W satisfying $W \not\supseteq L_0$.

  $\vdots$
- $X_{n-1} = \mathcal{T}(V)$

(c) For $i = 1, 2, \ldots n-2$, a vertex in $X_i \setminus X_{i-1}$ is a subspace W of dimension i not containing L. Describe $\mathrm{Lk}_{X_i}(W)$, and verify that the vertex $W \oplus L_0$ is a cone point of the link. (Where are we using the assumption that $i \leq n-2$?)

(d) The subspace X_0 is a cone (with vertex L_0) and thus is contractible. Apply Theorem I inductively to deduce that X_1, X_2, ..., and X_{n-2} are contractible.

(e) For a vertex W in $\mathcal{T}(V) \setminus X_{n-2}$, show that $\mathrm{Lk}_{\mathcal{T}(V)}(W) = \mathcal{T}(W)$.

(f) Use Theorem I and the inductive hypothesis on n to conclude Theorem II.

Exercise 6. (Bonus) Using the proof of Exercise 5 and induction on $\dim(V)$, describe a basis for $\widetilde{H}_{n-2}(\mathcal{T}(V))$. In the case that k is a finite field, find a formula for the rank of this group.

Exercise 7. (Bonus) Let $V \cong k^{2n}$ be a symplectic vector space with symplectic form ω. The Tits building $\mathcal{T}^{\omega}(V)$ is the order complex of the poset of isotropic subspaces of V under inclusion. Adapt the proof of Exercise 5 to prove that $\mathcal{T}^{\omega}(V)$ is $(n-2)$-connected (and hence a wedge of $(n-1)$-spheres). *Hint*: See Section 5.2 of Bestvina's notes.