1 "Badness" Arguments

This worksheet describes a standard tool used in the study of homological stability to prove that families of complexes are highly connected. It is sometimes called a "link" argument or a "badness" argument. We follow the exposition from Section 2.1 of Hatcher–Vogtmann "Tethers and homology stability for surfaces".

Theorem I. *Let Y be an abstract simplicial complex, and $X \subseteq Y$ a subcomplex. Fix $d \geq 0$. Suppose there is a set of "bad" simplices of $Y \setminus X$ satisfying the following conditions*

(i) If σ is a simplex such that no subsimplex $\rho \subseteq \sigma$ (including σ itself) is bad, then σ is in X.

(ii) If σ, ρ are bad simplices and $\sigma \cup \rho$ is a simplex, then $\sigma \cup \rho$ is bad.

(iii) For each bad simplex σ, the subcomplex $G_\sigma \subseteq \mathrm{Lk}_Y(\sigma)$ defined by

$$G_\sigma = \{\rho \in \mathrm{Lk}_Y(\sigma) \mid \text{ any bad subsimplex of } \rho \cup \sigma \text{ is contained in } \sigma\}$$

is $(d - dim(\sigma) - 1)$-connected.

Then the pair $(|Y|, |X|)$ is d-connected.

Recall this means that the inclusion $X \hookrightarrow Y$ induces isomorphisms on π_k for $k < d$ and surjection on π_d.

Corollary II. *Fix $d \geq 0$. Suppose $X \subseteq Y$ has a set of "bad" simplices satisfying the conditions of Theorem I.*

- *If $|X|$ is d-connected, then $|Y|$ is d-connected.*
- *If $|Y|$ is $(d-1)$-connected, then $|X|$ is $(d-1)$-connected.*

This result gives us two new strategies to prove that a simplicial complex is highly connected. We can attempt to find a highly connected subcomplex X, or attempt to embed the complex in a highly connected complex Y, in such a way that we can identify a suitable set of "bad" simplices.

Exercise 1. Deduce Corollary II from Theorem I.

Exercise 2. Verify that our statement of PL Morse Theory (Worksheet #18 Theorem I) is the special case of Theorem I and Corollary II where the "bad" simplices are vertices.

Exercise 3. (Bonus) Suppose that Y is a near-cone in the sense of Worksheet #15 Definition II, with associated subcomplex $X = \mathrm{Star}_Y(v_0)$ and additional simplices $B_{v_0}(Y)$. Show that the "bad" simplices $B_{v_0}(Y)$ satisfy the criteria of Theorem I if and only if all simplices in $B_{v_0}(Y)$ have dimension at least $(d+1)$. Reconcile this result with our description from Worksheet #15 of Y as a wedge of spheres.

Exercise 4. Fix integers $n \geq d \geq 1$. Let $A^{n,d}$ be the poset of proper subsets of $[n] = \{1, 2, \ldots, n\}$ of size at least d, ordered by inclusion. Prove that the order complex of $A^{n,d}$ is $(n-d-2)$-connected. *Hint:* Use a badness argument to show that the inclusion of $A^{n,d}$ into the barycentric subdivision of $\partial\Delta^{n-1}$ is highly connected.

Exercise 5. In this exercise, we will prove Theorem I. Suppose simplicial complexes $X \subseteq Y$ have a set of bad simplices satisfying the conditions of the theorem.

(a) Choose a vertex $x_0 \in X \subseteq Y$ as a basepoint. Let $0 \leq p \leq d$, and let $\mathbb{D}^p$ be the closed p-disk. Explain why the simplicial approximation theorem implies that a continuous map $(\mathbb{D}^p, \partial\mathbb{D}^p) \to (|Y|, x_0)$ can be represented (up to relative homotopy) as the realization of a simplicial map of abstract simplicial complexes $f : B^p \to Y$ for some triangulation B^p of $\mathbb{D}^p$.

(b) Let $0 \leq p \leq d$. Let $g : B^p \to Y$ be a simplicial map such that $g(\partial|B^p|) \subseteq |X|$. Our goal is to construct a homotopy rel $\partial|B^p|$ of the map $|g| : |B^p| \to |Y|$ so that its image lies in $|X|$. Explain why this would imply that the map $\iota_* : \pi_k(|X|, x_0) \to \pi_k(|Y|, x_0)$ induced by the inclusion $\iota : X \hookrightarrow Y$ is surjective for all $0 \leq k \leq d$ and injective for $0 \leq k \leq d-1$, and hence conclude the theorem.

(c) Call a simplex τ of B^p "bad" if $f(\tau) = \sigma$ is a bad simplex of Y. Explain why B^p can have at most finitely many bad simplices. In particular, we can choose a bad simplex τ_0 of B^p that is maximal in dimension. Let $\sigma_0 = f(\tau_0)$.

(d) Explain why maximality of τ_0 ensures that $f(\mathrm{Lk}_{B^p}(\tau_0)) \subseteq G_{\sigma_0}$.

(e) It is a result from PL topology that—since $|B^p|$ is locally Euclidean p-space—we can assume that $|\mathrm{Lk}_{Z^p}(\tau_0)|$ is homeomorphic to a sphere $S^{p-\dim(\tau_0)-1}$. Explain why the map $\mathrm{Lk}_{Z^p}(\tau_0) \to G_{\sigma_0}$ can be extended to a map $g : D^{p-\dim(\tau_0)} \to G_\sigma$ from a simplicial $(p - \dim(\tau_0))$-disk $D^{p-\dim(\tau_0)}$ with boundary $\mathrm{Lk}_{Z^p}(\tau_0)$.

(f) Show there is a well-defined map $|f|\Big|_{\partial\tau_0} \star |g|$, and that it is homotopic rel $\mathrm{Lk}_{B^p}(\tau_0)$ to the map $|f|\Big|_{\mathrm{Star}_{B^p}(\tau_0)}$. This homotopy is illustrated below in the case that τ_0 is 0 or 1 dimensional.

(g) Explain why we can extend this homotopy to a continuous homotopy of $|f|$ that is stationary on the complement of $|\mathrm{Star}_{B^p}(\tau_0)|$ in $|B^p|$. Verify in particular that this homotopy is stationary on $\partial|B^p|$.

(h) Deduce that—without changing the relative homotopy type of $|f|$—we can replace the simplicial structure on $\mathrm{Star}_{B^p}(\tau_0)$ by $\partial\tau_0 \star D^{p-\dim(\tau_0)}$ and redefine f on this subcomplex to be $f\Big|_{\partial\tau_0} \star g$.

(i) Verify we have modified f to remove a bad simplex of maximal dimension (possibly replacing it in part with a union of bad simplices of lower dimension), without introducing any new bad simplices of maximal dimension.

(j) Conclude that, by repeating this procedure a finite number of times, the resulting map will have image contained in X. This completes the proof.