1 Review of CW Complexes: Foundations

1.1 CW complexes

Recall that a CW complex is a topological space (sometimes with additional data of its CW structure) that is constructed inductively by attaching closed n-disks along their boundaries, in the following sense.

Definition I. (CW complexes). A *CW complex* X is a topological space constructed inductively as follows.

1. Its 0-skeleton $X^{(0)}$ is a discrete set of points, called the 0*-cells*.
2. The n-skeleton $X^{(n)}$ is constructed from the $(n-1)$-skeleton $X^{(n-1)}$ by attaching closed n-disks D^n_α along their boundaries via continuous *attaching maps* $\phi_\alpha : \partial D^n_\alpha \to X^{(n-1)}$. This means $X^{(n)}$ is the quotient space
$$q_n : X^{(n-1)} \sqcup_\alpha D^n_\alpha \longrightarrow X^{(n)}$$
defined by the equivalence relation generated by the identifications $x \sim \phi_\alpha(x)$ for all $x \in \partial D^n_\alpha$.
3. This process may stop after finitely many steps, in which case $X = X^{(n)}$ for some $n \in \mathbb{Z}_{\geq 0}$. Otherwise, we let $X = \cup_n X^{(n)}$, topologized with the *weak topology*: A set $A \subseteq X$ is open (respectively, closed) if and only if $A \cap X^{(n)}$ is open (respectively, closed) for all n.

We use the following terminology for CW complexes.

- For each (n, α), the composite $D^n_\alpha \hookrightarrow X^{(n-1)} \sqcup_\alpha D^n_\alpha \to X^{(n)} \to X$ is called the *characteristic map* Φ^n_α.
- The *(open) n-cell* e^n_α of X is the image of the interior of the disk D^n_α. The closure $\overline{e^n_\alpha}$ of e^n_α in X is called a *closed n-cell*.
- The *dimension* of a nonempty CW complex X is the maximal dimension of a cell in X. By convention we will say that the empty set has dimension -1.
- A CW complex is called *finite* if it is constructed from finitely many disks D^n_α.
- A *subcomplex* A of a CW complex X is a union of cells such that is closed in X. In particular, the closure of each cell is contained in A.

The following result follows from the definition of the weak topology.

Proposition II. *Let X be a CW complex, and Y a topological space. A map $f : X \to Y$ is continuous if and only if its restrictions $f\big|_{X^{(n)}}$ to each skeleton $X^{(n)}$ of X are continuous.*

Worksheet #1 Exercise 3 and induction over the skeleta of X imply the following proposition. In particular, the n-skeleton $X^{(n)}$ is a subcomplex of X.

Proposition III. *Let X be a CW complex with defining maps as in Definition I.*

- *The image $q_{n+1}(X^{(n)})$ is closed in $X^{(n+1)}$ and hence closed in X.*
- *The maps $X^{(n)} \to X^{(n+1)}$ and $X^{(n)} \to X$ are embeddings for all $n \geq 0$.*
- *The quotient map q_n and characteristic map Φ^n_α restrict to a homeomorphism from the interior of D^n_α to e^n_α.*
- *The cells e^n_α are disjoint and their union is X.*

Exercise 1. (Bonus) Let X be a CW complex and $A \subseteq X$ a subcomplex.

(a) Observe that the n-skeleton $X^{(n)}$ is itself a CW complex. Verify that the quotient topology defining $X^{(n)}$ agrees with its weak topology. Verify this topology agrees with its topology as a subspace of X.

(b) Verify $A \subseteq X$ inherits a CW complex structure, and its weak topology agrees with the subspace topology.

We will see that the topology on X is also the weak topology on X when viewed as a union of closed cells.

Proposition IV. *Let X be a CW complex with characteristic maps Φ^n_α. Then $\bigsqcup_{n,\alpha} \Phi^n_\alpha : \bigsqcup_{n,\alpha} D^n_\alpha \longrightarrow X$ is a quotient map.*

The following criteria for continuity of maps are a defining feature of CW complexes.

Corollary V. *Let X be a CW complex and Y a topological space.*

- *A subset C of X is closed if and only if it intersects each closed cell in a closed set.*
- *A map $f : X \to Y$ is continuous if and only if its restrictions to each closed cell of X are continuous.*
- *A homotopy of maps $F : I \times X \to Y$ is continuous if and only if its restrictions $F_{I \times \overline{e^n_\alpha}}$ to each closed cell $\overline{e^n_\alpha}$ are continuous.*

Exercise 2. Let X be a CW complex.

(i) Prove Proposition IV.

(ii) Deduce Corollary V. *Hint:* Recall Worksheet # 1 Exercise 2.

Corollary VI. *Let X be a finite CW complex with attaching maps ϕ^n_α. Let $* = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_N = X$ be a filtration on X by sub-CW-complexes such that $F_{i+1} \setminus F_i$ is a single (open) cell $e^{n_i}_{\alpha_i}$. Then the subcomplexes F_i, and X, are the quotient spaces constructed inductively by attaching $D^{n_i}_{\alpha_i}$ to F_i via the attaching map $\phi^{n_i}_{\alpha_i}$.*

Exercise 3. Outline a strategy to prove Corollary VI. You do not need to check details.

CW complexes have the following (mostly favourable) point-set properties.

Proposition VII. *Let X be a CW complex.*

- *X is Hausdorff.*
- *X is normal; disjoint closed subsets have disjoint open neighbourhoods.*
- *X is locally path-connected.*
- *X is connected if and only if it is path-connected.*
- *The closure $\overline{e^n_\alpha}$ of the open n-cell e^n_α in X (or in $X^{(n)}$) is equal to the image $\Phi^n_\alpha(D^n_\alpha)$.*
- *The characteristic map Φ^n_α is a quotient map onto its image.*
- *X is compact if and only if it is finite.*
- *Any compact subset C of X is contained in a finite subcomplex. In particular, each closed cell of X intersects finitely many (open) cells.*
- *X is locally compact if and only if it is locally finite (each point is contained in finitely many closed cells).*
- *X is locally contractible, i.e., it has a basis of open subsets that are (as subspaces) contractible.*
- *Any subcomplex $A \subseteq X$ has a neighbourhood U_A in X that deformation retracts to A. For subcomplexes $A, B \in X$, these neighbourhoods can be constructed in such a way that $U_{A\cap B} = U_A \cap U_B$.*
- *If X is a CW complex and $A \subseteq X$ a subcomplex, the quotient space $X \to X/A$ inherits a CW complex structure.*
- *There are homeomorphisms of cells $e^m \times e^n \to e^{m+n}$. Consequently, the product of CW complexes X and Y inherits the structure of a CW complex. In general the weak topology does not agree with the product topology. However, they do agree if either X or Y is locally compact. The topologies agree if X and Y both have at most countably many cells.*
- *Any covering space of X inherits a CW complex structure.*

In fact, the following theorem gives an equivalent definition of a CW complex. This was Whitehead's original definition, and it explains the terminology "CW" ("closure-finite, weak topology").

Theorem VIII. *Let X be a space and $\Phi^n_\alpha : D^n_\alpha \to X$ a family of maps. Then X is a CW complex with characteristic maps $\Phi^n_\alpha : D^n_\alpha \to X$ if and only if the following conditions hold.*

- *X is Hausdorff.*
- *Φ^n_α maps $\mathrm{Int}(D^n_\alpha)$ homeomorphically onto its image $e^n_\alpha := \Phi^n_\alpha(\mathrm{Int}(D^n_\alpha))$.*
- *The cells e^n_α are disjoint and their union is X.*
- **(closure-finite)** *Φ^n_α maps ∂D^n_α to a finite union of cells of dimension less than n.*
- **(weak topology)** *A subset is closed in X if and only if it intersects each closed cell $\overline{e^n_\alpha}$ in a closed set.*

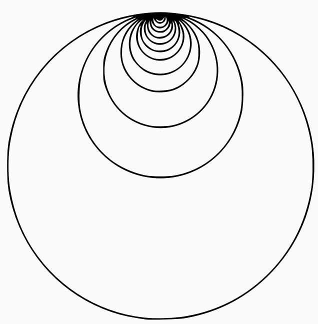
The closure-finite condition ensures that each closed cell only intersects finitely many open cells.

Exercise 4. Working directly from (i) Definition I and (ii) Theorem VIII, explain why the following are not valid CW complex structures.

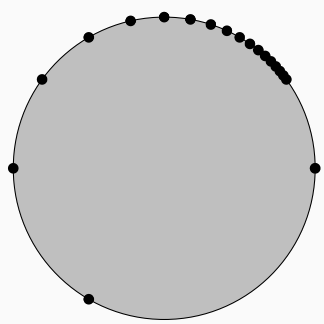
(a) The closed interval I as an uncountable union of 0-cells.

(b) The infinite earring, as a union of one 0-cell and countably infinite 1-cells.

(c) A closed 2-disk, with one 2-cell and countably infinite 0- and 1-cells in its boundary.



(b) The infinite earring



(c) A 2-disk with countably many cells in its boundary

Exercise 5. (Bonus) Prove that the infinite earring is not homotopy equivalent to a CW complex.

Exercise 6. (Bonus) Prove Propositions VII and VIII.