1 Discrete Morse Theory

This worksheet is based on Forman's notes "A user's guide to discrete Morse theory". See also Kozlov Ch. 11. The goal is the following: given a simplicial complex X, construct a CW complex with fewer cells but the same homotopy type.

Definition I. Let X be a finite abstract simplicial complex with vertex set $V(X)$ and simplices $S(X)$. Given a function $f : S(X) \to \mathbb{R}$, define for each simplex α the following quantities,

$$F_{f,\alpha} = \#\{\tau \mid \tau \subseteq \alpha \text{ is a codimension-1 face, and } f(\tau) \geq f(\alpha)\}$$

$$C_{f,\alpha} = \#\{\sigma \mid \alpha \subseteq \sigma \text{ is a codimension-1 face, and } f(\alpha) \geq f(\sigma)\}.$$

A function $f : S(X) \to \mathbb{R}$ is a *Morse* function on X if $F_{f,\alpha}$ and $C_{f,\alpha}$ are at most 1 for all simplices α of X.

The quantities $F_{f,\alpha}, C_{f,\alpha}$ in a sense measure the failure of f to be strictly order-preserving with respect to the poset structure of $S(X)$.

Definition II. Let X be an abstract simplicial complex and $f : S(X) \to \mathbb{R}$ a Morse function. We call a simplex α *critical* if $F_{f,\alpha} = C_{f,\alpha} = 0$, and we call $f(\sigma)$ a *critical value* of f.

Exercise 1. Let X be an abstract simplicial complex and f a discrete Morse function for X. Show that, for each simplex α of X, at most one of $F_{f,\alpha}$ and $C_{f,\alpha}$ can be nonzero.

Exercise 2. Let X be a finite abstract simplicial complex, and $f : S(X) \to \mathbb{R}$ a Morse function. Explain how to modify f so that it is injective but remains a Morse function with the same set of critical simplices. We may therefore assume (without loss of generality) that our Morse functions are injective.

Definition III. Let X be a finite abstract simplicial complex, and f a discrete Morse function. For $c \in \mathbb{R}$, define

$$M(c) = \bigcup_{f(\beta) \leq c} \bigcup_{\alpha \subseteq \beta} \alpha$$

that is, $M(c)$ is the simplicial closure of the preimage $f^{-1}(-\infty, c]$.

Theorem IV (Forman)**.** *Let X be a finite abstract simplicial complex, and $f : S(X) \to \mathbb{R}$ a Morse function.*

- *If the interval $[a, b]$ contains no critical values of f, then $M(b)$ deformation retracts to $M(a)$.*
- *Suppose σ is a critical p-simplex with $f(\sigma) \in [a, b]$, and no other critical simplices have values in $[a, b]$. Then $M(b)$ is homotopy equivalent to a space*

  $$M(a) \bigcup_{\partial D^p} D^p$$

  obtained by gluing a p-cell D^p to $M(a)$ along its boundary.

Corollary V. *Let X be a finite abstract simplicial complex, and f a discrete Morse function on X. Then X is homotopy equivalent to a CW complex with precisely one p-cell for every critical p-simplex of X.*

In fact, from the data of the Morse function, we can explicitly write the cellular chain complex for this CW complex, which (by the Corollary) has homology equal to $H_*(X)$. See Forman Section 7.

Corollary VI. *Let X be a finite abstract simplicial complex, and f a discrete Morse function on X. Suppose that the only critical simplices of f are one 0-cell, and cells of dimension p. Then X is homotopy equivalent to a wedge of p-spheres.*

Exercise 3. (Bonus) Let X be a finite abstract simplicial complex. Label every vertex of X by a nonnegative number, and every simplex by the sum of its vertices. Show that this defines a Morse function on X where every simplex is critical, and that Theorem IV is vacuous in this case. What if we label vertices by positive and negative numbers?

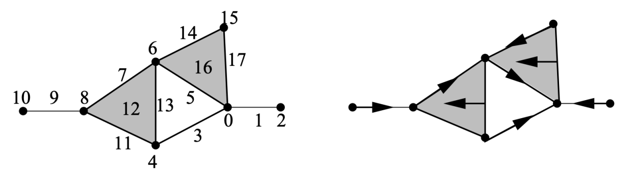
Exercise 4. Deduce Corollary VI from Corollary VI.

1.1 Discrete vector fields

It follows from Theorem IV that the key information encoded by a Morse function f on X is the set of critical simplices of X, equivalently, the set of noncritical simplices. By Exercise 1, the noncritical simplices form a disjoint set of pairs $\alpha \subseteq \beta$, where α is comdimension-1 face of β such that $f(\alpha) > f(\beta)$.

These pairs can be represented geometrically: for each noncritical pair $\alpha \subseteq \beta$, draw an arrow from the barycentre of α to the barycentre of β. These arrows should be thought of as representing (the negative of) a gradient vector field corresponding to f.

This figure (from Forman) shows a discrete Morse function and the corresponding vector field.



In practice, to apply Theorem IV, it can be more practical to construct a vector field than a Morse function; below is a combinatorial criterion that ensures a "discrete" vector field arises from a valid Morse function.

Definition VII. Let X be an abstract simplicial complex. A *discrete vector field* V on X is a collection of pairs $\{\alpha, \beta\}$ of simplices of X such that $\alpha \subseteq \beta$ is a codimension-1 face, and such that each simplex is in at most one pair of V.

Theorem VIII (Forman)**.** *A discrete vector field V is the gradient vector field of a discrete Morse function if and only if there are no* non-trivial closed V-paths*; i.e., there are no finite sequences*

$$\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_r, \beta_r, \alpha_{r+1} = \alpha_0$$

such that $\{\alpha_i, \beta_i\}$ is a pair in V, $\alpha_{i+1} \subseteq \beta_i$ is a subsimplex, $\alpha_i \neq \alpha_{i+1}$, and $r \geq 0$.

Exercise 5. (Bonus) Verify that the discrete Morse function in the above figure is, in fact, a valid Morse function, and that the discrete vector field shown is the associated gradient vector field. Verify the conclusions of Theorem IV and its corollaries in this example.

Exercise 6. (Bonus) Prove Theorem VIII. *Hint:* See Forman Theorem 3.5 and Section 6.

Exercise 7. Let $0 \leq k \leq n$. Let $X^{n,k} = (\Delta^n)^{(k)}$ be the k-skeleton of an n-simplex. Use discrete Morse theory to re-calculate the homotopy type of $X^{n,k}$. *Hint:* Consider a discrete vector field corresponding to pairs of simplices of the form $(\sigma \setminus \{1\}, \sigma \cup \{1\})$ for all simplices $\sigma \subseteq [n+1]$ for which these two sets are simplices of $X^{n,k}$.

1.2 An application: the complex of partitions of n

Definition IX. Fix an integer $n \geq 1$. An *(unordered) partition of* n is a tuple of integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. A component λ_i is called a *part* of the partition λ.

Note that the collection of partitions of the number n can be viewed as the orbits (under the symmetric group action) of the collection of partitions of the set $[n]$.

Definition X. Let $n \geq 1$, and let Λ_n be the poset of partitions of n, ordered by refinement. Let $\overline{\Lambda}_n$ denote the subposet that excludes the coarsest partition $\lambda = (n)$ and the finest partition $\lambda = (1, 1, \ldots, 1)$.

Theorem XI. *For $n \geq 3$ the order complex $|\overline{\Lambda}_n|$ of the poset of partitions of n is contractible.*

Exercise 8. In this exercise we will prove Theorem XI, following Kozlov Theorem 11.19. Call a partition λ *special* if every part has size 1 or 2.

(a) Our goal is to construct a discrete Morse function for which the critical simplices are precisely the simplices spanned by special partitions. Explain why the existence of this discrete Morse function will imply the theorem.

(b) For a partition λ of n, define the partition $\mu_2(\lambda)$ to be the refinement of λ that replaces each even part λ_i with the parts of the partition $(2, 2, \ldots, 2)$ of λ_i, and replaces every odd part λ_i with the parts of the partition $(2, 2, \ldots, 2, 1)$ of λ_i. Verify that $\mu_2(\lambda) = \lambda$ if and only if λ is special.

(c) Define a discrete vector field on $\overline{\Lambda}_n$ as follows. For a simplex σ, suppose σ has at least one non-special vertex. Let $\lambda(\sigma)$ denote the finest partition contained in σ that is not special. Consider the pairing $(\sigma \setminus \{\mu_2(\lambda(\sigma))\}, \sigma \cup \{\mu_2(\lambda(\sigma))\})$. Verify that this defines a partition of the noncritical simplices into disjoint sets of size exactly 2.

(d) Complete the proof by verifying that this discrete vector field satisfies the condition of Theorem VIII. *Hint:* Consider the number of special vertices in each simplex of a hypothetical nontrivial closed V-path.

1.3 An application: the complex of disconnected graphs on $[n]$

Forman gives the following application of discrete Morse theory.

Definition XII. Let $[n] := \{1, 2, 3, \ldots, n\}$. Let P_n be the set of the $\binom{n}{2}$ unordered pairs in $[n]$. For any graph G with vertex set $[n]$, we can encode G as a subset of $E(G)$ of P, where $E(G)$ contains the pair $\{i, j\}$ if and only if G has an edge between vertices i and j. Let $\mathcal{N}_n$ be the abstract simplicial complex with vertex set P_n, and for which $\sigma \subseteq P_n$ is a simplex precisely when σ corresponds to a graph on vertex set $[n]$ that has at least one edge and is disconnected.

Following Forman, we will prove the following result using discrete Morse theory.

Theorem XIII. *The complex $|\mathcal{N}_n|$ is homotopy equivalent to the wedge of $(n-1)!$ many spheres of dimension $(n-3)$.*

Exercise 9. (Bonus) Verify that $\mathcal{N}_n$ is a valid simplicial complex, i.e., the faces of its simplices are simplices.

Exercise 10. (Bonus) Prove Theorem XIII. *Hint:* See Forman Section 5.