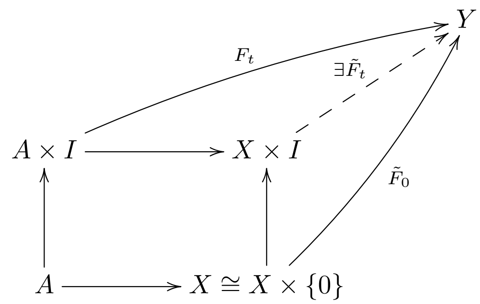
1 Review of CW Complexes: The homotopy extension property

Definition I. Let X be a topological space and $A \subseteq X$ a subspace. Let $I = [0,1]$ denote the closed interval. We say that the pair (X, A) has the *homotopy extension property* if, given a homotopy $F_t(a)$ from $A \times I \to Y$ and a map $\tilde{F}_0 : X \to Y$ such that $\tilde{F}_0|_A = F_0$, then there is a homotopy $\tilde{F}_t(x)$ from $X \to Y$ such that $\tilde{F}_t|_A = F_t$. The homotopy$\tilde{F}_t(x)$ is called an *extension* of $F_t(a)$. The lift $\tilde{F}_t(x)$ need not be unique.



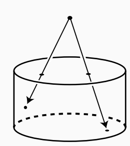
Theorem II. *Let A be a subcomplex of a CW complex X.*

(i) The space $X \times I$ deformation retracts onto $(X \times \{0\}) \cup (A \times I)$.

(ii) The pair (X, A) has the homotopy extension property.

Exercise 1. (Bonus) In this (optional) exercise, we outline a proof of Theorem II.

(a) Describe a deformation retraction from $D^n \times I$ to $(D^n \times \{0\}) \cup (\partial D^n \times I)$. The following image (from Hatcher) suggests one construction.



(b) Use part (a) to construct a deformation retraction $H^{(n)}$ from $X^{(n)} \times I$ onto $(X^{(n)} \times \{0\}) \cup \Big((X^{(n-1)} \cup A^{(n)}) \times I \Big)$.

(c) Deduce that we can construct the desired deformation retraction from $X \times I$ onto $(X \times \{0\}) \cup (A \times I)$ by concatenating these homotopies, performing $H^{(n)}$ during the time interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. Verify that this homotopy is continuous on $X \times I$. This concludes Theorem II part (i).

(d) Show Theorem II part (i) implies part (ii).

Exercise 2. (Bonus) Verify that the pair $(\mathbb{R}, \mathbb{Q})$ does not have the homotopy extension property.

Proposition III (See Hatcher *Algebraic Topology* Proposition 0.18)**.** *Let A be a subcomplex of a CW complex X. Given homotopic attaching maps $f, f' : A \to Y$, there is a homotopy equivalence $X \sqcup_f Y \simeq X \sqcup_{f'} Y$ via homotopies that restricts to the identity on Y at all times.*

In fact,

Theorem IV (Lundell–Weingram *The Topology of CW complexes*, Ch IV Theorem 2.3)**.** *Let X be a CW complex and $A \subseteq X$ a subcomplex. Suppose $h : Y \to Y'$ is a homotopy equivalence of topological spaces. Let $f : A \to Y$, $f' A \to Y'$ be continuous maps. If f' is homotopic to hf, then h extends to a homotopy equivalence $H : X \sqcup_f Y \to X \sqcup_{f'} Y'$. Any homotopy inverse g of h extends to a homotopy inverse of H.*

Of particular interest is the case that $X = D^n$ and $A = \partial D^n$, so the quotient space is the space obtained by attaching an n-cell to Y.

Corollary V. *Let Y be a space, and let $\phi_1, \phi_2 : \partial D^n \to Y$ be continuous maps. If $\phi_1 \simeq \phi_2$, then the space $Y \sqcup_{\phi_1} D^n$ obtained by gluing an n-cell onto Y along ϕ_1 is homotopy equivalent rel Y to the space $Y \sqcup_{\phi_2} D^n$ obtained by gluing an n-cell onto Y along ϕ_2.*

Corollary VI. *The homotopy type of a CW complex depends only on the homotopy classes of its attaching maps.*

Corollary VII. *Let X be a CW complex and $D \subseteq X$ a CW subcomplex. If D is contractible, then the quotient map $X \to X/D$ is a homotopy equivalence.*

Exercise 3. (Bonus) Show by example that the quotient of a topological space by a contractible subspace may not be a homotopy equivalence in general. *Hint:* Consider $(S^1 \setminus \{x_0\}) \subseteq S^1$.

Given a space A, recall that the cone CA on A is the quotient space $CA = (A \times I)/(A \times \{1\})$. We view A as a subspace of CA by identifying it with the image of $A \times \{0\}$. (Check this is an embedding!) The (unreduced) suspension SA of A is the quotient CA/A.

Corollary VIII. *Let X be a CW complex and $\iota : A \to X$ be the inclusion of a subcomplex. Then $X/A \simeq X \sqcup_\iota CA$.*

Corollary IX. *Let X be a CW complex and $\iota : A \to X$ the inclusion of a subcomplex. If ι is nullhomotopic, then $X/A \simeq X \vee SA$.*

The following special cases of these results will be the key input to our combinatorial tools such as the concept of shellability.

Corollary X. *Let Y be a space and let $D^n \sqcup_\phi Y$ be the space obtained by attaching an n-disk D^n along its boundary via the map $\phi : \partial D^n \to Y$. If ϕ is nullhomotopic, then $D^n \sqcup_\phi Y$ is homotopy equivalent to the wedge product $Y \vee S^n$.*

Corollary XI. *Let Y be CW complex. Let A be a CW complex such that $\iota_Y : A \hookrightarrow Y$ and $\iota_D : A \hookrightarrow D^n$ are inclusions of A as CW subcomplexes of Y and the n-disk D^n. If A is contractible, then $D^n \sqcup_{\iota_Y} Y$ is homotopy equivalent to Y.*

Exercise 4. Assuming Theorem IV, deduce the corollaries.

Exercise 5. Prove Proposition III: Let $F : A \times I \to Y$ be a homotopy from f to f'. Verify that $(X \times I) \sqcup_F Y$ deformation retracts onto both $X \sqcup_f Y$ and $X \sqcup_{f'} Y$.

Exercise 6. (Bonus) The purpose of this exercise is to prove Theorem IV.

(a) Show Proposition III reduces Theorem IV to the case $h \circ f = f'$.

(b) Show that the composite

$$X \sqcup Y \xrightarrow{id_X \sqcup h} X \sqcup Y' \xrightarrow{\text{glue}} X \sqcup_{f'} Y'$$

factors through a map $H : X \sqcup_f Y \to X \sqcup_{f'} Y'$.

(c) Consider the mapping cylinder of H. Show that it deformation retracts to both $X \sqcup_{f'} Y'$ and $X \sqcup_f Y$. See Lundell–Weingram Theorem 2.3 for details.